5.5 The Substitution Rule

$\int f(u)\,du$ du is a "differential"

$\boxed{u = g(x)}$ $\boxed{du = g'(x)\,dx}$

Leibniz notation $du = \frac{du}{dx}\,dx$

Recall $\circledast$ $\int F'(x)\,dx = F(x) + C$

What if $F(x) = f(g(x))$?

$F'(x) = f'(g(x))\,g'(x)$ by the chain rule

Thus, by $\circledast$,

$$\int f'(g(x))\,g'(x)\,dx = f(g(x)) + C$$

This is the substitution rule but to simplify the notation we set $u = g(x)$ $du = g'(x)\,dx$

$$\int f'(u)\,du = f(u) + C$$

Example $\int \underbrace{2x}_{g'(x)} \sqrt{\underbrace{1+x^2}_{g(x)}}\,dx \left(= \int \sqrt{g(x)}\,g'(x)\,dx \quad \text{where } g(x) = 1+x^2\right)$

Set $u = 1+x^2$ $du = 2x\,dx$

$$\int \sqrt{1+x^2}\,2x\,dx = \int \sqrt{u}\,du = \int u^{1/2}\,du = \frac{u^{3/2}}{3/2} + C$$

$$= \frac{2}{3}u^{3/2} + C = \frac{2}{3}(1+x^2)^{3/2} + C$$

Example $\int \sqrt{2x+1}\,dx$ $u = 2x+1$ $du = 2\,dx$

$\frac{1}{2}du = dx$

$$= \frac{1}{2}\int \sqrt{u}\,du = \frac{1}{2}\int u^{1/2}\,du$$

$$= \frac{1}{2}\,\frac{u^{3/2}}{\frac{3}{2}} + C = \frac{1}{\cancel{2}}\,\frac{\cancel{2}}{3}u^{3/2} + C = \frac{1}{3}(2x+1)^{3/2} + C$$

Example $\int \frac{x\,dx}{\sqrt{1-4x^2}}$ $u = 1-4x^2$ $du = -8x\,dx$

$-\frac{1}{8}du = x\,dx$

$$= -\frac{1}{8}\int \frac{1}{\sqrt{u}}\,du = -\frac{1}{8}\int u^{-1/2}\,du$$

$$= -\frac{1}{8}\,\frac{u^{1/2}}{\frac{1}{2}} + C = -\frac{1}{4}u^{1/2} + C = -\frac{1}{4}\sqrt{1-4x^2} + C$$

Example $\int e^{5x}\,dx$ $u = 5x$ $du = 5\,dx$

$\frac{1}{5}du = dx$

$$= \frac{1}{5}\int e^u\,du = \frac{1}{5}e^u + C = \frac{1}{5}e^{5x} + C$$

(Rule: $\int e^{kx}\,dx = \frac{1}{k}e^{kx} + C$)

Example $\int \sqrt{1+x^2}\,x^5\,dx$ $u = 1+x^2$ $du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$$= \int \sqrt{1+x^2}\,x^4\,x\,dx$$

$x^2 = u-1 \longrightarrow x^4 = (u-1)^2$

$$= \frac{1}{2}\int \sqrt{u}\,(u-1)^2\,du$$

$$= \frac{1}{2}\int u^{1/2}(u^2-2u+1)\,du = \frac{1}{2}\int (u^{5/2} - 2u^{3/2} + u^{1/2})\,du$$

$$= \frac{1}{2}\left[\frac{u^{7/2}}{\frac{7}{2}} - 2\,\frac{u^{5/2}}{\frac{5}{2}} + \frac{u^{3/2}}{\frac{3}{2}}\right] + C = \frac{1}{2}\left(\frac{2}{7}u^{7/2} - \frac{4}{5}u^{5/2} + \frac{2}{3}u^{3/2}\right) + C$$

$$= \frac{1}{7}(1+x^2)^{7/2} - \frac{2}{5}(1+x^2)^{5/2} + \frac{1}{3}(1+x^2)^{3/2} + C$$

Example $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$

"Wrong" $u = \sin x$ $du = \cos x\,dx$

However $\cos x$ is in the denominator!

Correct $u = \cos x$ $du = -\sin x\,dx$

$$\int \frac{\sin x}{\cos x}\,dx = -\int \frac{du}{u} = -\int \frac{1}{u}\,du = -\ln|u| + C$$

$$= -\ln|\cos x| + C = \ln|\cos x|^{-1} + C = \ln\left|\frac{1}{\cos x}\right| + C$$

$$= \ln|\sec x| + C$$

Example We have seen that $\int \sqrt{2x+1}\,dx = \frac{1}{3}(2x+1)^{3/2} + C$

Thus,

$$\int_0^4 \sqrt{2x+1}\,dx = \frac{1}{3}(2x+1)^{3/2}\Big|_0^4 = \frac{1}{3}(9)^{3/2} - \frac{1}{3}(1)^{3/2}$$

$$= \frac{1}{3}3^3 - \frac{1}{3} = \frac{26}{3}$$

The substitution rule for definite Integrals:

$$\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

Example $\displaystyle\int_{x=0}^{x=4} \sqrt{2x+1}\,dx$ $\quad u = 2x+1 \quad du = 2\,dx$

$\frac{1}{2}du = dx$

When $x=0$, $u = 2x+1 = 1$ (lower limit)

when $x=4$, $u = 2x+1 = 9$ (upper limit)

$$= \frac{1}{2}\int_1^9 \sqrt{u}\,du = \frac{1}{\cancel{2}}\frac{\cancel{2}u^{3/2}}{3}\Big|_1^9 = \frac{1}{3}u^{3/2}\Big|_1^9 = \frac{1}{3}9^{3/2} - \frac{1}{3}1^{3/2}$$

$$= \frac{26}{3}$$

Example $\displaystyle\int_1^2 \frac{dx}{(3-5x)^2}$ $\quad u = 3-5x \quad du = -5\,dx$

$-\frac{1}{5}du = dx$

when $x=2$, $u = 3-10 = -7$ (upper limit)

when $x=1$, $u = 3-5 = -2$ (lower limit)

$$= -\frac{1}{5}\int_{-2}^{-7}\frac{du}{u^2} = -\frac{1}{5}\int_{-2}^{-7} u^{-2}\,du = -\frac{1}{5}\,\frac{u^{-1}}{-1}\Big|_{-2}^{-7}$$

$$= \frac{1}{5}\cdot\frac{1}{u}\Big|_{-2}^{-7} = \frac{1}{5}\left(-\frac{1}{7}\right) - \frac{1}{5}\left(-\frac{1}{2}\right) = -\frac{1}{35} + \frac{1}{10}$$

$$= \frac{1}{10} - \frac{1}{35}$$

Example Find $\displaystyle\int_1^e \frac{\ln x}{x}\,dx$ $\quad u = \ln x \quad du = \frac{1}{x}dx$

$u = \ln(e) = 1$ (upper)

$u = \ln(1) = 0$ (lower)

$$= \int_0^1 u\,du = \frac{u^2}{2}\Big|_0^1 = \frac{1}{2}$$

Symmetry (Integrals of symmetric functions)

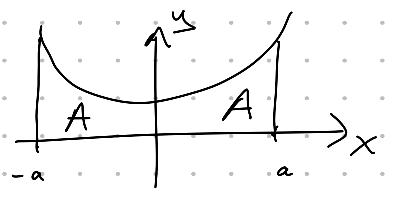
f is even if $f(x) = f(-x)$ e.g. $x^4 + x^2 + \cos x$

f is odd if $-f(x) = f(-x)$ e.g. $x^3 + 5x + \sin x$

$(-x^3 = (-x)^3)$

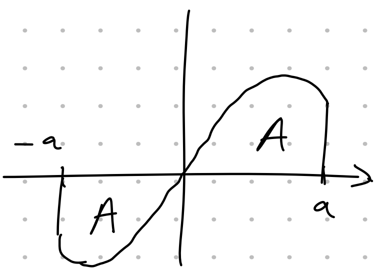
If f is even then

$$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx = 2A$$



If f is odd then

$$\int_{-a}^{a} f(x)\,dx = 0 = A - A$$



Example $\displaystyle\int_{-2}^{2}(x^6+1)\,dx$ $\quad$ note that $f(x) = x^6+1 = (-x)^6+1$

$= f(-x)$

so f is even.

$$\int_{-2}^{2}(x^6+1)\,dx = 2\int_0^2 (x^6+1)\,dx = 2\left(\frac{x^7}{7}+x\right)\Big|_0^2 = 2\left(\frac{2^7}{7}+2\right)$$

$$= \frac{2^8}{7} + 4$$

Example $\displaystyle\int_{-1}^{1}\frac{\tan x}{1+x^2+x^4}\,dx$ $\quad f(x) = \frac{\tan x}{1+x^2+x^4}$

$$f(-x) = \frac{\tan(-x)}{1+(-x)^2+(-x)^4} = \frac{-\tan x}{1+x^2+x^4} = -f(x)$$

f is odd thus $\displaystyle\int_{-1}^{1} f(x)\,dx = 0$ ✓

$$\tan(-x) = \frac{\sin(-x)}{\cos(-x)} = \frac{-\sin(x)}{\cos(x)} = -\tan x$$