Last time:

**FTC2**: If $F$ is an antiderivative of $f$ $(F'=f)$, then $\boxed{f \text{ cont. on } [a,b]}$

$$\int_a^b f(x)\,dx = F(b)-F(a)$$

or $$\int_a^b F'(x)\,dx = F(b)-F(a) = F(x)\Big|_a^b$$

Notation: $f(x)\Big|_a^b = f(b)-f(a)$ $\qquad f\Big|_a^b = f(b)-f(a)$

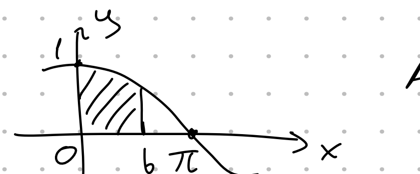
$$f\Big|_b^a = f(a)-f(b) = -(f(b)-f(a)) = -f\Big|_a^b$$

**Example** Evaluate $\displaystyle\int_3^6 \frac{dx}{x} = \int_3^6 \frac{1}{x}\,dx$

$(\ln|x|+C)' = \frac{1}{x}$ since $3\le x\le 6$, $(\ln(x))' = \frac{1}{x}$

$$\int_3^6 \frac{1}{x}\,dx = \ln(6)-\ln(3)$$

**Example** Find the area under the curve $y=\cos(x)$ from $0$ to $b$ where $0\le b\le \frac{\pi}{2}$.




$$\text{Area} = \int_0^b \cos(x)\,dx = \sin x\Big|_0^b = \sin b - \sin 0$$
$$= \sin b \qquad (\text{since } (\sin x)' = \cos x)$$

**Example** What is wrong with the following calculation?

$$\int_{-1}^3 \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_{-1}^3 = -\frac{1}{3} - \left(\frac{+1}{+1}\right) \qquad x^{-2} = \left(\frac{x^{-1}}{-1}\right)' = (-x^{-1})'$$
$$-x^{-1} = -\frac{1}{x}$$
$$= -\frac{1}{3} - 1 = -\frac{4}{3} < 0$$

$\frac{1}{x^2} \ge 0$ always above the $x$-axis so the fact that we got a negative number is a sign that something went wrong.

$\frac{1}{x^2}$ is not cont. on $[-1,3]$. It has a vert. asymp. at $x=0$.

## 5.4 Indefinite Integrals and the Net Change Theorem

Indefinite Integrals = antiderivatives

Notation: $\displaystyle\int f(x)\,dx = F(x)$ means that $F(x)$ is an antiderivative (or an indefinite integral) of $f(x)$. In other words $F'(x) = f(x)$.

e.g. $\displaystyle\int x^2\,dx = \frac{x^3}{3} + C$

**Note that**: $\displaystyle\int_a^b f(x)\,dx$ is just a number.

$\displaystyle\int f(x)\,dx$ is a function of $x$

We can restate FTC2 using this notation:
$$\int_a^b f(x)\,dx = \int f(x)\,dx\Big|_a^b$$

A few rules of indefinite integrals:

① $\displaystyle\int c f(x)\,dx = c\int f(x)\,dx$ $\qquad$ ② $\displaystyle\int (f(x)\pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

③ $\displaystyle\int k\,dx = kx + C$

④ $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \qquad (n\ne -1)$

⑤ $\displaystyle\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln|x| + C$

⑥ $\displaystyle\int e^x\,dx = e^x + C$

**Example** Find $\displaystyle\int (10x^4 - 2\sec^2 x)\,dx$

$$= 10\int x^4\,dx - 2\int \sec^2 x\,dx = 10\,\frac{x^5}{5} - 2\tan x + C$$
$$= 2x^5 - 2\tan x + C$$

**Example** Evaluate $\displaystyle\int_0^3 (x^3-6x)\,dx = \frac{x^4}{4} - \frac{6x^2}{2}\Big|_0^3$

$$= \frac{x^4}{4} - 3x^2\Big|_0^3 = \frac{3^4}{4} - 3(3^2) - (0-0) = \frac{81}{4} - 27$$

**Example** Find $\displaystyle\int_0^2 \left(2x^3 - 6x + \frac{3}{x^2+1}\right)dx$

$$= \left(\frac{2x^4}{4} - \frac{6x^2}{2}\right)\Bigg|_0^2 + 3\int_0^2 \frac{1}{1+x^2}\,dx = \frac{32}{4} - 3(4) - 0 + 3\tan^{-1}x\,\Big]_0^2$$

$$= 8 - 12 + 3\tan^{-1}(2) - 3\underbrace{\tan^{-1}(0)}_{=0} = -4 + 3\tan^{-1}(2)$$

Example Find $\displaystyle\int_1^9 \frac{2t^2 + t^2\sqrt{t} - 1}{t^2}\,dt$

$$= \int_1^9 \left(\frac{2t^2}{t^2} + \frac{t^2\sqrt{t}}{t^2} - \frac{1}{t^2}\right)dt = \int_1^9 \left(2 + t^{1/2} - t^{-2}\right)dt$$

$$= 2t + \frac{t^{3/2}}{\frac{3}{2}} - \frac{t^{-1}}{-1}\Bigg|_1^9 = 2t + \frac{2}{3}t^{3/2} + \frac{1}{t}\Bigg|_1^9$$

$$= 2(9) + \frac{2}{3}\,9^{3/2} + \frac{1}{9} - \left(2 + \frac{2}{3} + 1\right) = 18 + \frac{2}{\cancel{3}}(\cancel{27}^{\,9}) + \frac{1}{9} - \left(3 + \frac{2}{3}\right)$$

$$\boxed{9^{3/2} = \left(9^{1/2}\right)^3 = (3)^3 = 27} \qquad = 36 + \frac{1}{9} - 3 - \frac{2}{3} = 33 - \frac{6}{9} + \frac{1}{9}$$

$$= 33 - \frac{5}{9}$$

## The Net Change Theorem

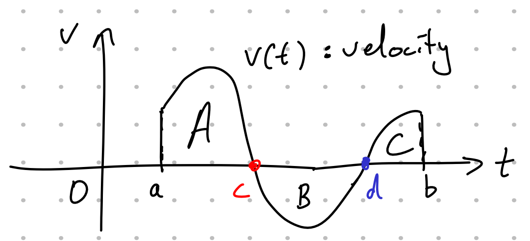
The integral of a rate of change is the net change:

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

e.g. If $F(t)$ is the population of a city at time $t$, its rate of change $F'(t)$ measures how fast the population is changing and $\int_a^b F'(x)\,dx = F(b) - F(a)$ : the net change in population

e.g. If $s(t)$ represents the position of an object that moves along a straight line and $v(t) = s'(t)$ is its velocity then $\int_a^b v(t)\,dt = s(b) - s(a)$ : the net change in position or in other words displacement.

displacement $\neq$ total distance traveled.




Q) How to get the total distance traveled?

A) total distance traveled

$$= A + B + C$$

$$= \int_a^c v(t)\,dt - \int_c^d v(t)\,dt + \int_d^b v(t)\,dt$$

$$= \int_a^c |v(t)|\,dt + \int_c^d |v(t)|\,dt + \int_d^b |v(t)|\,dt$$

$$= \int_a^b |v(t)|\,dt = \text{total distance traveled}$$

$A, B, C > 0$ because they represent the corresponding

$$\text{Displacement} = \int_a^b v(t)\,dt = A - B + C$$

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Example A particle moves with velocity function

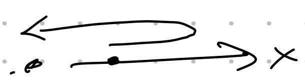
$v(t) = t^2 - t - 6$ (m/s)

a) Find the displacement of the particle for $1 \le t \le 4$

b) Find the total distance traveled for $1 \le t \le 4$.

(a) $$\int_1^4 v(t)\,dt = \frac{t^3}{3} - \frac{t^2}{2} - 6t\,\Bigg|_1^4 = \frac{4^3}{3} - \frac{4^2}{2} - 6(4) - \left(\frac{1}{3} - \frac{1}{2} - 6\right)$$

$$= -\frac{9}{2}$$



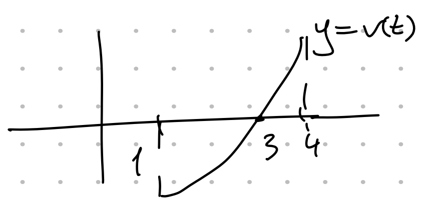

(b) $$\int_1^4 |v(t)|\,dt$$

$v(t) = t^2 - t - 6 = (t-3)(t+2)$

for $1 \le t \le 3$ $v(t)$ negative

for $3 \le t \le 4$ $v(t)$ positive

$$= -\int_1^3 v(t)\,dt + \int_3^4 v(t)\,dt$$




$$= -\left(\frac{t^3}{3} - \frac{t^2}{2} - 6t\right)\Bigg|_1^3 + \left(\frac{t^3}{3} - \frac{t^2}{2} - 6t\right)\Bigg|_3^4$$

$$= \underline{-\left(\frac{3^3}{3} - \frac{3^2}{2} - 6(3)\right) + \left(\frac{1}{3} - \frac{1}{2} - 6(1)\right)} + \left(\frac{4^3}{3} - \frac{4^2}{2} - 6(4)\right) \underline{- \left(\frac{3^3}{3} - \frac{3^2}{2} - 6(3)\right)}$$

$$= \frac{61}{6}$$