Last time:

Definition The definite integral of f from a to b is

$$\underline{\int_a^b f(x)\,dx} = \lim_{n\to\infty} \underline{\left(f(x_1^*)+f(x_2^*)+\cdots+f(x_n^*)\right)\Delta x} = \lim_{n\to\infty} \underline{\sum_{i=1}^{n} f(x_i^*)\Delta x}$$

$\uparrow$ new notation!

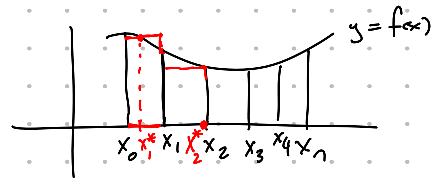
$$\underset{\text{integral sign}}{\int}_{\underset{\uparrow \text{lower limit}}{a}}^{b \leftarrow \text{upper limit}} \underbrace{f(x)}_{\text{integrand}}\,d\underbrace{x}_{\text{"integration variable is x"}}$$

"integral of f from a to b"

- The process of computing $\int_a^b f(x)dx = \underline{\lim_{n\to\infty} \sum_{i=0}^{n} f(x_i^*)d_x}$ is called integration.

- If the <u>limit</u> exists, f is called integrable on $[a,b]$.

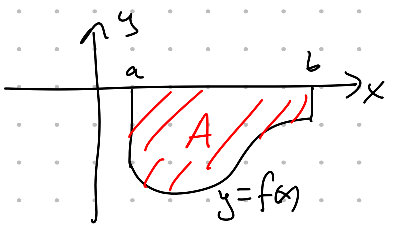
$\sum_{i=1}^{n} f(x_i^*)\,dx$ is called Riemann sum.



$x_i^* \in [x_{i-1}, \underline{x_i}]$
$\uparrow$
sample point.

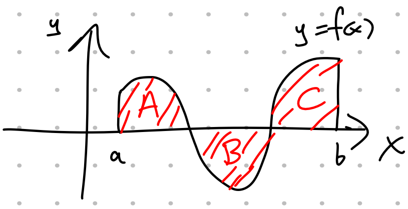
- is a "Right Hand" Riemann sum if $x_i^* = x_i$ (or Right hand Sum)
- is a "Left Hand" Riemann sum if $x_i^* = x_{i-1}$
- give us the midpoint rule if $x_i^* = \frac{x_{i-1}+x_i}{2}$

Note that $\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$ may make sense even if $f(x) \le 0$ for all x or if f(x) is sometimes above the x-axis and sometimes below!



<u>A is the area of the shaded region</u> (so A is positive)

$$\int_a^b f(x)dx = -A < 0$$



$$\int_a^b f(x)dx = A - B + C$$

the area below the x-axis counts <u>negatively.</u>

That is, $\int_a^b f(x)dx =$ "<u>the net area</u>"

<u>Thm</u> If f is cont. on $[a,b]$, or if it has only finitely many jump discontinuities, then $\int_a^b f(x)d_x$ exists, or f is integrable.

<u>Thm</u> If f is integrable on $[a,b]$ then

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \quad (\text{Right Hand Sum})$$

where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$.

Evaluate Integrals

$$1+2+3+\cdots+n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$1^2+2^2+3^2+\cdots+n^2 = \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$1^3+2^3+3^3+\cdots+n^3 = \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

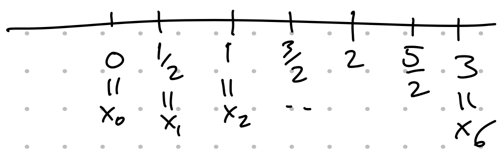
<u>Properties of Σ:</u> ① $\sum_{i=1}^{n} c = c + c + \cdots + c = nc$

② $\sum_{i=1}^{n} c a_i = c \sum_{i=1}^{n} a_i$ ③ $\sum_{i=1}^{n} (a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$

<u>Example</u> a) Evaluate the Riemann sum for $f(x) = x^3 - 6x$, taking the sample points to be right endpoints and $a=0$, $b=3$, and $n=6$

b) Evaluate $\int_0^3 (x^3 - 6x)dx$.

ⓐ $\Delta x = \frac{b-a}{n} = \frac{3-0}{6} = \frac{1}{2}$



$x_0 = a = 0$
$x_1 = a + \Delta x = \frac{1}{2}$
$x_2 = a + 2\Delta x = 1$
$\vdots$
$x_6 = a + 6\Delta x = 3$

$$\text{Riemann sum} = \sum_{i=1}^{6} f(x_i)\Delta x$$

$$= \left(f(\tfrac{1}{2}) + f(1) + f(\tfrac{3}{2}) + f(2) + f(\tfrac{5}{2}) + f(3)\right)\Delta x$$

$f(x) = x^3 - 6x$

$$= \left((\tfrac{1}{2})^3 - 6(\tfrac{1}{2}) + (1^3 - 6(1)) + \cdots + (3^3 - 6(3))\right)\frac{1}{2}$$

$$= -3.9375$$

ⓑ $\Delta x = \frac{b-a}{n} = \frac{3-0}{n} = \frac{3}{n}$ $f(x) = x^3 - 6x$

$x_1 = a + \Delta x = \frac{3}{n}$
$x_2 = a + 2\Delta x = 2(\frac{3}{n})$
$x_3 = a + 3\Delta x = 3(\frac{3}{n})$
$\vdots$
$x_n = a + n\Delta x = n(\frac{3}{n})$
$x_i = a + i\Delta x = i(\frac{3}{n})$

$$\sum_{i=1}^{n} f(x_i)\Delta x = \underline{\left(f(\tfrac{3}{n}) + f(2(\tfrac{3}{n})) + \cdots (f(n(\tfrac{3}{n}))\right)\Delta x}$$

$$\parallel$$

$$\sum_{i=1}^{n} f\left(\frac{3i}{n}\right)\cdot\frac{3}{n} = \frac{3}{n}\sum_{i=1}^{n}\left[\frac{3^3 i^3}{n^3} - 6\left(\frac{3i}{n}\right)\right]$$

$$= \frac{3}{n}\left(\sum_{i=1}^{n} \frac{27 i^3}{n^3} - \sum_{i=1}^{n} \frac{18 i}{n}\right)$$

$$= \frac{3}{n}\left[\frac{27}{n^3}\underbrace{\sum_{i=1}^{n} i^3}_{\downarrow} - \frac{18}{n}\underbrace{\sum_{i=1}^{n} i}_{\downarrow}\right]$$

using formulas for $\sum_{i=1}^{n} i$ and $\sum_{i=1}^{n} i^3$

$$= \frac{3}{n}\left[\frac{27}{n^3}\left(\frac{n(n+1)}{2}\right)^2 - \frac{\overset{9}{\cancel{18}}}{\cancel{n}}\left(\frac{\cancel{n}(n+1)}{\underset{1}{\cancel{2}}}\right)\right]$$

$\sum_{i=1}^{n} f(x_i)\Delta x$ ||

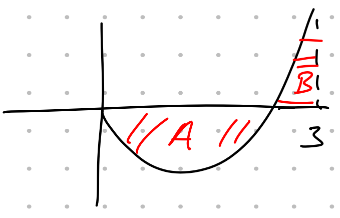
$$= \frac{3}{n}\left(\frac{27}{n^{\cancel{3}}}\left(\frac{\cancel{n^2}(n+1)^2}{4}\right) - 9(n+1)\right) = \frac{3}{n}\cdot\frac{27(n+1)^2 - 36n(n+1)}{4n}$$

Next we take $\underline{\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x}$

$$= \lim_{n\to\infty}\frac{81(n+1)^2 - 108n(n+1)}{4n^2} = \lim_{n\to\infty}\frac{81\left(\frac{n+1}{n}\right)^2 - 108\frac{\cancel{n}}{\cancel{n}}\left(\frac{n+1}{n}\right)}{4\frac{\cancel{n^2}}{\cancel{n^2}}}$$

$$= \lim_{n\to\infty}\frac{81\left(1+\frac{1}{n}\right)^2 - 108\left(1+\frac{1}{n}\right)}{4} = \frac{81-108}{4} = -6.75$$

Thus, $\int_0^3 (x^3-6x)\,dx = -6.75 = -A+B$



Example

Set up an expression for $\int_1^3 e^x\,dx$ as a limit of sums.

$\Delta x = \frac{b-a}{n} = \frac{3-1}{n} = \frac{2}{n}$ $\qquad f(x) = e^x$

$x_i = a + i\Delta x = 1 + \frac{2i}{n}$

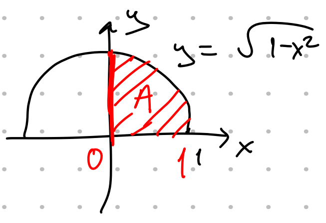
$$\int_1^3 e^x\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty}\sum_{i=1}^{n} e^{\left(1+\frac{2i}{n}\right)}\frac{2}{n}$$

Example

Evaluate the following integrals by interpreting each in terms of areas.

a) $\int_0^1 \sqrt{1-x^2}\,dx$ $\qquad$ b) $\int_0^3 (x-1)\,dx = -A+B$
$= -\frac{1}{2}+2 = \frac{3}{2}$

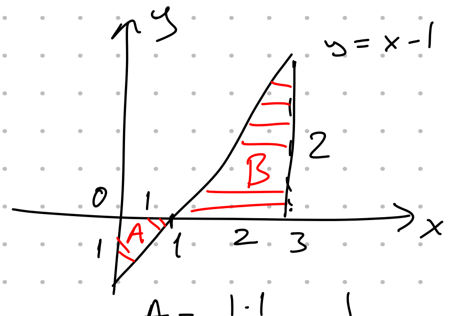
$y = \sqrt{1-x^2}$
$\hookrightarrow y^2 = 1-x^2 \rightarrow x^2+y^2=1$



$\underline{\underline{r=1}}$

$A = \frac{1}{4}\pi r^2 = \frac{\pi}{4}$

$\int_0^1 \sqrt{1-x^2}\,dx = \frac{\pi}{4}$

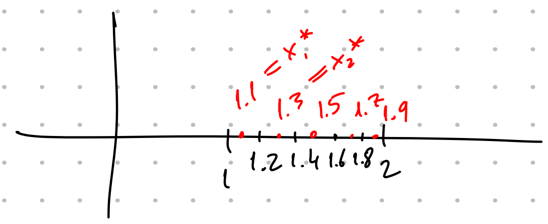


$A = \frac{1\cdot 1}{2} = \frac{1}{2}$ ✓

$B = \frac{2\cdot 2}{2} = 2$ ✓

Example

Use the $\underline{\text{midpoint rule}}$ with $\underline{n=5}$ to approximate $\int_1^2 \frac{1}{x}\,dx$



$$\sum_{i=1}^{5} f(x_i^*)\Delta x$$

$$= \left(\frac{1}{1.1}+\frac{1}{1.3}+\frac{1}{1.5}+\frac{1}{1.7}+\frac{1}{1.9}\right)0.2$$

$$\approx 0.691908$$

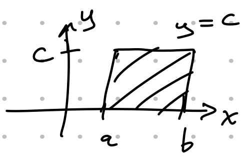
$\Delta x = \frac{b-a}{n} = \frac{2-1}{5} = \frac{1}{5} = 0.2$

Properties of the Definite Integrals

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx \qquad \underline{\Delta x} = \frac{b-a}{n} = -\frac{a-b}{n}$$

In particular, $(b=a)$ $\int_a^a f(x)\,dx = 0$

① $\int_a^b c\,dx = c(b-a)$



② $\int_a^b [f(x)\pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

③ $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$ $\qquad$ where c is any constant.