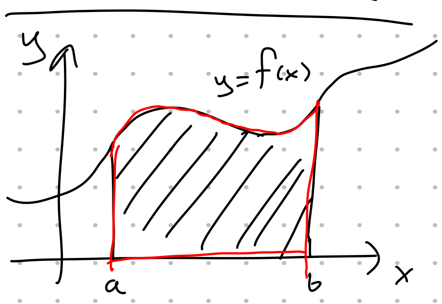
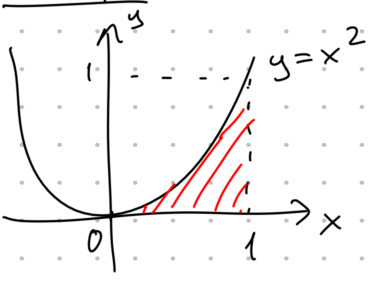
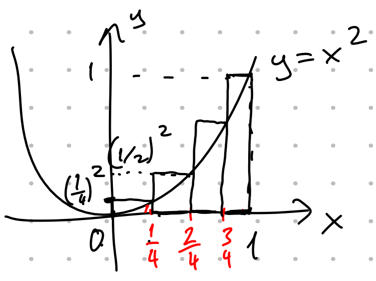
The Area Problem



What is the area of the region bounded by $x=a, x=b$, $y=f(x)$ and the x-axis?

Example

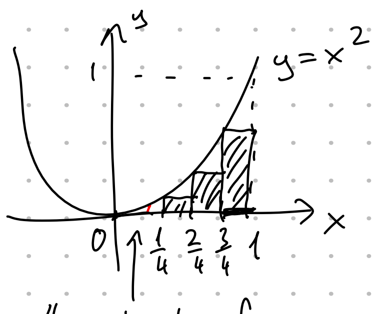




Area $\approx$ sum of the areas of the rectangles $= \underset{b}{\frac{1}{4}}\underset{h}{\left(\frac{1}{4}\right)^2} + \underset{b}{\frac{1}{4}}\underset{h}{\left(\frac{1}{2}\right)^2} + \underset{b}{\frac{1}{4}}\underset{h}{\left(\frac{3}{4}\right)^2} + \underset{b}{\frac{1}{4}}\underset{h}{(1)^2}$

R_4 because the top Right corner of the rectangles touch the curve and we are using 4 rectangles.

$R_4 = \frac{1}{4}\left(\frac{1}{4}\right)^2 + \frac{1}{4}\left(\frac{1}{2}\right)^2 + \frac{1}{4}\left(\frac{3}{4}\right)^2 + \frac{1}{4}(1)^2 = \frac{15}{32}$

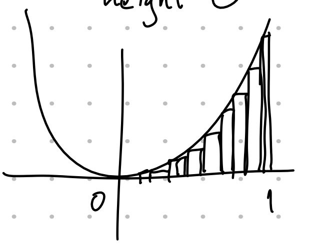


"rectangle of height 0"

left

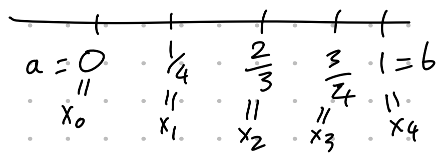
Area $\approx L_4 = \underset{b}{\frac{1}{4}}\underset{h}{(0)^2} + \underset{b}{\frac{1}{4}}\underset{h}{\left(\frac{1}{4}\right)^2} + \underset{b}{\frac{1}{4}}\underset{h}{\left(\frac{1}{2}\right)^2} + \underset{b}{\frac{1}{4}}\underset{h}{\left(\frac{3}{4}\right)^2}$

$= \frac{7}{32}$



If we increase the number of rectangles, both R_n and L_n is going to approach to the actual area

Notation:



(n: number of rectangles)
$n=4$

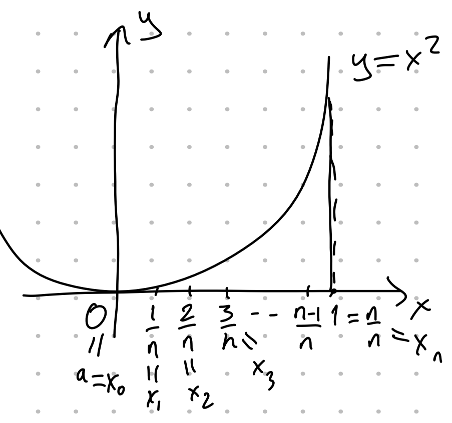
Δx: base of rectangles
$\Delta x = \frac{1}{4}$

$R_n = R_4 = \Delta x\left(f(x_1)+f(x_2)+f(x_3)+f(x_4)\right)$

$L_n = L_4 = \Delta x\left(f(x_0)+f(x_1)+f(x_2)+f(x_3)\right)$

Example Find the area between the x-axis and the curve $y=x^2$ for $0 \le x \le 1$ precisely.

We will use n rectangles to find R_n. Then we will take the limit $\lim_{n\to\infty} R_n$.



$\Delta x = \frac{1}{n}$ $\qquad f(x)=x^2$

$R_n = \Delta x\left(f(x_1)+f(x_2)+f(x_3)+\dots+f(x_n)\right)$

$= \frac{1}{n}\left(\left(\frac{1}{n}\right)^2+\left(\frac{2}{n}\right)^2+\left(\frac{3}{n}\right)^2+\dots\left(\frac{n}{n}\right)^2\right)$

$= \frac{1}{n}\left(\frac{1^2}{n^2}+\frac{2^2}{n^2}+\frac{3^2}{n^2}+\dots+\frac{n^2}{n^2}\right)$

$= \frac{1}{n^3}\left(1^2+2^2+3^2+\dots+n^2\right)$

$\boxed{1^2+2^2+3^2+\dots+n^2 = \frac{n(n+1)(2n+1)}{6}}$ $\qquad = \frac{\cancel{n}(n+1)(2n+1)}{6\,n^{\cancel{3}\,2}}$

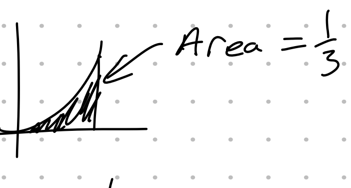
$R_n = \frac{(n+1)(2n+1)}{6n^2}$

$\lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{\frac{(n+1)}{n}\frac{(2n+1)}{n}}{6\cancel{n^2}}$

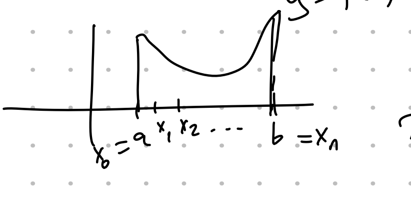
$= \frac{1}{6}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) = \frac{1}{6}\cdot(2) = \frac{1}{3}$

You may check that

$\lim_{n\to\infty} L_n = \lim_{n\to\infty} \Delta x\left(f(x_0)+f(x_1)+\dots+f(x_{n-1})\right) = \frac{1}{3}$



In general:



n = number of rectangles

$\Delta x = \frac{b-a}{n}$

Then we have subintervals: $[x_0,x_1], [x_1,x_2], \dots [x_{n-1},x_n]$

$x_0 = a \qquad x_n = b$

$x_1 = a+\Delta x$
$x_2 = a+2\Delta x$
$x_3 = a+3\Delta x$
$\vdots$
$x_i = a+i\Delta x$
$\vdots$
$x_n = a+n\Delta x$

$R_n = \left(f(x_1)+f(x_2)+f(x_3)+\dots+f(x_n)\right)\Delta x$

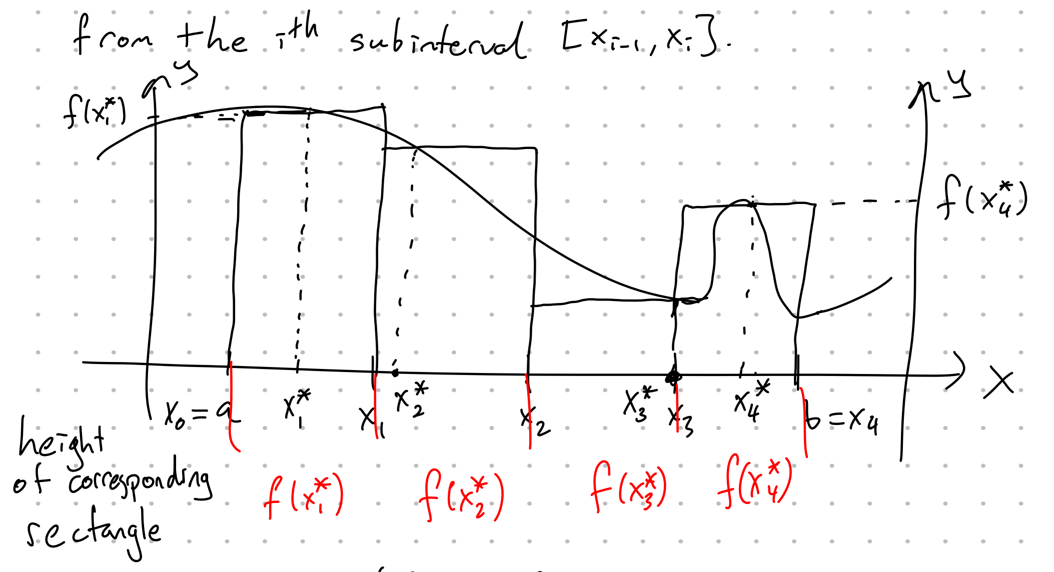
Definition The area of the region bounded by $x=a$, $x=b$, the x-axis and $y=f(x)$ is given by

$\text{Area} = \lim_{n\to\infty} R_n$

We will get the same area if we use "left hand sum"

$$\text{Area} = \lim_{n\to\infty} R_n = \lim_{n\to\infty} L_n$$

In fact, more generally, we can pick any "sample point" x_i^* from the ith subinterval $[x_{i-1}, x_i]$.

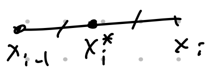


$$A = \lim_{n\to\infty} \left(f(x_1^*) + f(x_2^*) + f(x_3^*) + \cdots + f(x_n^*)\right)\Delta x$$

"Right hand sum" is a special case where $x_i^* = x_i$

"Left hand sum" is a special case where $x_i^* = x_{i-1}$

"Midpoint rule" is a special case where $x_i^* = \frac{x_{i-1}+x_i}{2}$



Sigma notation

$$R_4 = \sum_{i=1}^{4} f(x_i)\Delta x = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x$$

We plug in numbers $i=1, i=2, i=3, i=4$ in $f(x_i)\Delta x$ and add them.

$$\boxed{\sum_{i=m}^{n} f(i) = f(m) + f(m+1) + f(m+2) + \cdots + f(n-1) + f(n)}$$

e.g. $1^2+2^2+3^2+\cdots+n^2 = \sum_{i=1}^{n} i^2 = \boxed{\frac{n(n+1)(2n+1)}{6}}$ formula we saw earlier

Example Let A be the area under the curve $y = e^{-x}$ for $0 \le x \le 2$.

a) Find an expression for A as a limit. (Do not evaluate)

b) Estimate the area by taking the sample points to be midpoints and using 4 subintervals.

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \left(f(x_1) + f(x_2) + f(x_3) + \cdots + f(x_n)\right)\Delta x$$

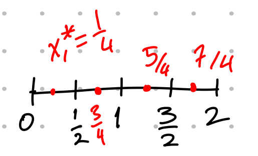
$f(x) = e^{-x}$ $\quad \Delta x = \frac{b-a}{n} = \frac{2-0}{n} = \frac{2}{n}$

$x_1 = a + \Delta x = 0 + \frac{2}{n} = \frac{2}{n}$, $\quad x_2 = a + 2\Delta x = \frac{4}{n}$, $x_3 = \frac{6}{n}, \ldots$

$x_n = a + n\Delta x = 0 + n\frac{2}{n} = \frac{2n}{n}$

$$A = \lim_{n\to\infty} \left(e^{-2/n} + e^{-4/n} + e^{-6/n} + \cdots + e^{-2n/n}\right)\frac{2}{n}$$

b



$$A \approx \left(f(x_1^*) + f(x_2^*) + f(x_3^*) + f(x_4^*)\right)\Delta x$$

$$= \left(e^{-1/4} + e^{-3/4} + e^{-5/4} + e^{-7/4}\right)\frac{2}{4}$$

$$\approx 0.8557$$

Example Suppose the odometer on our car is broken and we want to estimate the distance driven over a 30-second time interval. We take speedometer readings every five seconds:

Time (s)	0	5	10	15	20	25	30
Velocity (ft/s)	25	31	35	43	47	45	41

Over the first 5 seconds the distance $\approx 5s\left(25\,\frac{ft}{s}\right) = 125ft$

Over the second 5 seconds the distance $\approx 5s\left(31\,\frac{ft}{s}\right) = 155ft$

Over the first 5 seconds the distance $\approx 5s\left(35\,\frac{ft}{s}\right)$

⋮

$5\times25 + 5\times31 + 5\times35 + 5\times43 + 5\times47 + 5\times45 = 1130ft$

Over the first 5 seconds the distance $\approx 5s\left(31\,\frac{ft}{s}\right) = 155ft$

⋮

$5\times31 + 5\times35 + 5\times43 + 5\times47 + 5\times45 + 5\times41 = 1210ft$

These processes are exactly the same as finding area!

5.2 The Definite Integral

Definition The definite integral of f from a to b is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\left(f(x_1^*) + f(x_2^*) + \cdots + f(x_n^*)\right)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

new notation!