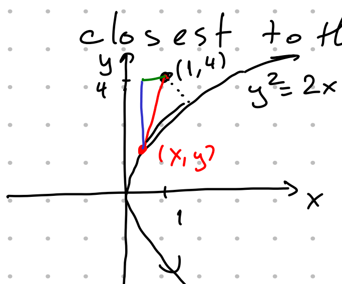
Example Find the point on the parabola $y^2 = 2x$ that is closest to the point $(1,4)$.



$$d = \sqrt{(x-1)^2 + (y-4)^2}$$

$$y^2 = 2x \implies x = \frac{y^2}{2}$$

$$d(y) = \sqrt{\left(\frac{y^2}{2}-1\right)^2 + (y-4)^2}$$

Want to minimize $d(y)$.

Instead we can minimize $\underbrace{(d(y))^2}_{D(y)=} = \left(\frac{y^2}{2}-1\right)^2 + (y-4)^2$

(This only works because $d(y) \geq 0$ for all y)

$$D'(y) = \frac{dD(y)}{dy} = 2\left(\frac{y^2}{2}-1\right)\cdot\left(\frac{\cancel{2}y}{\cancel{2}}\right) + 2(y-4)$$

$$= (y^2-2)y + 2y - 8 = y^3 - \cancel{2y} + \cancel{2y} - 8 = y^3 - 8 = 0$$

$y = 2$

y		2	
D'	$-$		$+$
D	dec		inc

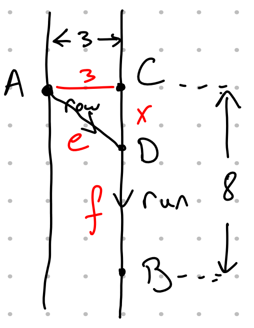
2 is a global min.

$y = 2$

$x = \frac{y^2}{2} = \frac{2^2}{2} = 2$

$(2,2)$

Example A man launches his boat from point A on a bank of a straight river, 3km wide, and wants to reach point B, 8km downstream on the opposite bank, as quickly as possible. He rows to some point D on the opposite bank.



If he can row 6 km/h and run 8 km/h, where should he land to reach B as soon as possible? (Neglect the speed of water.)

Want to minimize the total time.

Total time = rowing time + running time

$\frac{\text{Dist (km)}}{\text{Speed (km/h)}} = \text{Time (h)}$

$$= \frac{e}{6} + \frac{f}{8}$$

$e = \sqrt{x^2+3^2} = \sqrt{x^2+9}$ $\qquad f = 8-x$

$$\text{Total Time} = T(x) = \frac{\sqrt{x^2+9}}{6} + \frac{8-x}{8}$$

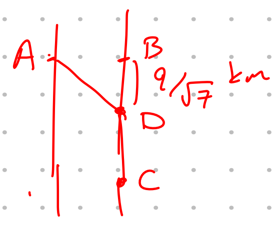
$$T'(x) = \frac{1}{6}\cdot\frac{1}{\cancel{2}}\cdot(x^2+9)^{-1/2}\cdot(\cancel{2}x) - \frac{1}{8} = \frac{x}{6\sqrt{x^2+9}}_{(4)} - \frac{1}{8}_{(3\sqrt{x^2+9})}$$

$$= \frac{4x - 3\sqrt{x^2+9}}{24\sqrt{x^2+9}} = 0 \implies 4x - 3\sqrt{x^2+9} = 0$$

$$4x = 3\sqrt{x^2+9}$$

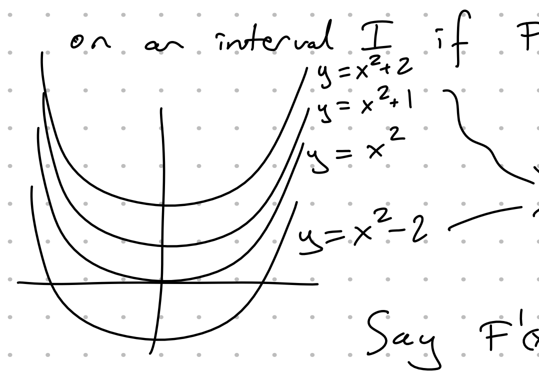
$$16x^2 = 9(x^2+9) = 9x^2 + 81 \implies 7x^2 = 81$$

$$x^2 = \frac{81}{7} \implies x = \frac{9}{\sqrt{7}} \text{ (km)}$$



4.9 Antiderivatives

Definition A function F is called an antiderivative of f on an interval I if $F'(x) = f(x)$ for all $x \in I$.



$y' = 2x$ they all have the same derivative.

Say $F'(x) = f(x) = G'(x)$

Then $F(x) = G(x) + C$

Theorem If F is an antiderivative of f on I, then the most general antiderivative of f on I is

$$F(x) + C$$

where C is an arbitrary constant.

Example Find the most general antiderivative of

a) $f(x) = \sin x$

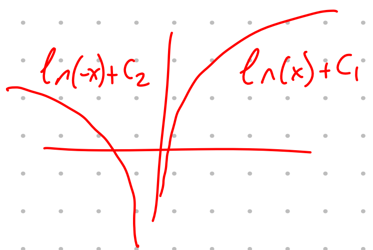
$F(x) = -\cos x + C$ $\qquad F'(x) = \sin x$

b) $f(x) = \frac{1}{x}$ $\qquad$ Domain $(-\infty, 0) \cup (0, \infty)$

We know that $(\ln x)' = \frac{1}{x}$ (for $x > 0$)

$(\ln -x)' = \frac{1}{(-x)}(-1) = \frac{1}{x}$ (for $x < 0$)

So
$$F(x) = \begin{cases} \ln x + C_1 & \text{if } x > 0 \\ \ln(-x) + C_2 & \text{if } x < 0 \end{cases}$$



$F(x) = \ln|x| + C$ ~~X~~

c) $f(x) = x^n \quad n \neq -1$ $\qquad (x^{n+1})' = (n+1)\,x^n$

$$\left(\frac{x^{n+1}}{n+1}\right)' = \frac{1}{n+1}(x^{n+1})' = x^n$$

$$F(x) = \frac{x^{n+1}}{n+1} + C$$

Example Find f if $f'(x) = e^x + 20(1+x^2)^{-1}$ and $f(0) = -2$.

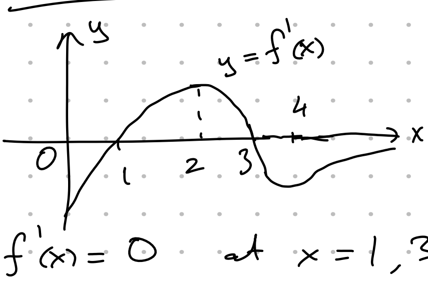
$(e^x)' = e^x \qquad (1+x^2)^{-1} = \frac{1}{1+x^2} = (\tan^{-1}x)'$

$f(x) = e^x + 20\tan^{-1}x + C \quad = e^x + 20\tan^{-1}(x) - 3$

$f(0) = e^0 + 20\tan^{-1}0 + C = -2$

$1 + C = -2 \qquad C = -3$

Example



Given the graph of $f'(x)$ and $f(0) = 2$

Sketch $f(x)$.

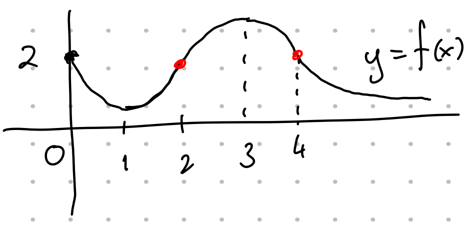
$f'(x) = 0$ at $x = 1, 3$

x		1		3	
$f'(x)$	$-$		$+$		$-$
$f(x)$	dec		inc		dec

local min at $x=1$ local max at 3

X	0	2		4	
f'	inc		dec		inc
f''	$+$		$-$		$+$
f	conc up		conc. down		conc up

inflection points



Exaple A particle moves in a straight line and has acceleration given by $a(t) = 6t + 4$. Its initial velocity is $v(0) = -6$ cm/s and its initial displacement is $s(0) = 9$ cm

Find its position function $s(t)$.

$(\text{velocity})' = \text{acceleration}$

$v(t) = 6\frac{t^2}{2} + 4t + C = 3t^2 + 4t + C$

$v(0) = 0 + 0 + C = -6$

$\Rightarrow v(t) = 3t^2 + 4t - 6$

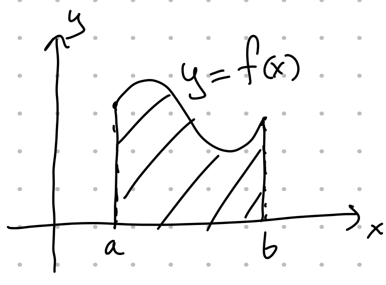
$(\text{position})' = \text{velocity}$

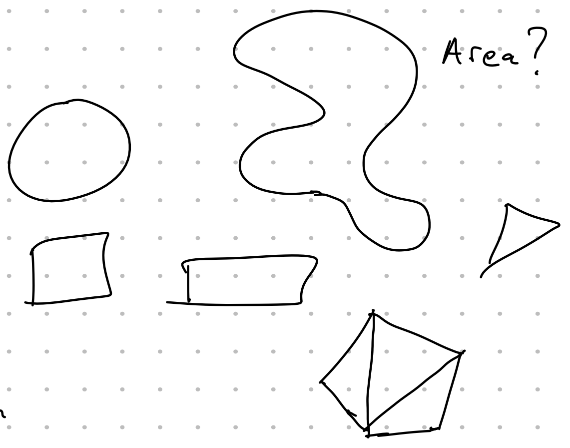
$s(t) = t^3 + 2t^2 - 6t + C$

$s(0) = 0 + 0 + 0 + C = 9 \Rightarrow s(t) = t^3 + 2t^2 - 6t + 9$

Chp 5 Integrals

5.1 Areas and Distances





What is the area between $y = f(x)$ and x-axis for $a \leq x \leq b$