Example Sketch $y = \frac{2x^2}{x^2-1} = f(x)$

A. Domain $x^2 - 1 \neq 0 \iff x \neq \pm 1$

$D = (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$

B. $y = f(0) = 0$ is the y-intercept.

$f(x) = 0 \iff \frac{2x^2}{x^2-1} = 0 \iff 2x^2 = 0 \iff x = 0$

$x = 0$ is the only x-intercept.

C. $f(-x) = \frac{2(-x)^2}{(-x)^2 - 1} = \frac{2x^2}{x^2-1} = f(x)$ even. symmetric about y-axis

D. $\lim_{x \to \pm\infty} \frac{2x^2}{x^2-1} = \lim_{x \to \pm\infty} \frac{2}{1 - \frac{1}{x^2}} = 2$ is the only hor. asymp.

$\lim_{x \to 1^+} \frac{2x^2}{x^2-1} = \infty$ $\quad \lim_{x \to 1^-} \frac{2x^2}{x^2-1} = -\infty$

$\lim_{x \to -1^+} \frac{2x^2}{x^2-1} = -\infty$ $\quad \lim_{x \to -1^-} \frac{2x^2}{x^2-1} = +\infty$

$-1 \quad 0$ $\qquad -1 \quad 0$

E. $f(x) = \frac{2x^2}{x^2-1}$ $\quad f'(x) = \frac{4x(x^2-1) - 2x^2(2x)}{(x^2-1)^2}$

$= \frac{\cancel{4x^3} - 4x - \cancel{4x^3}}{(x^2-1)^2} = -\frac{4x}{(x^2-1)^2} \longrightarrow$ always positive

So if $x > 0$, $f'(x) < 0$

if $x < 0$, $f'(x) > 0$

Inc on $(-\infty, -1) \cup (-1, 0)$

Dec on $(0, 1) \cup (1, \infty)$

F.

x		-1		0		1	
$f'(x)$	$+$		$+$		$-$		$-$
f	inc		inc		dec		dec

0 is a local max

G. $f'(x) = -\frac{4x}{(x^2-1)^2}$ so $f''(x) = -\frac{4(x^2-1)^2 - 4x(2(x^2-1)2x)}{(x^2-1)^4}$

$f''(x) = -\frac{4\cancel{(x^2-1)}[(x^2-1) - 4x^2]}{(x^2-1)^{\cancel{4}\,3}} = 4\left(\frac{4x^2 - (x^2-1)}{(x^2-1)^3}\right) = 4\frac{(3x^2+1)}{(x^2-1)^3}$

$3x^2+1$ is always positive

Thus, if $x > 1$ or $x < -1$, then $x^2 - 1 > 0$

$f''(x) > 0$

if $-1 < x < 1$, then $x^2 - 1 < 0$ $\quad f''(x) < 0$

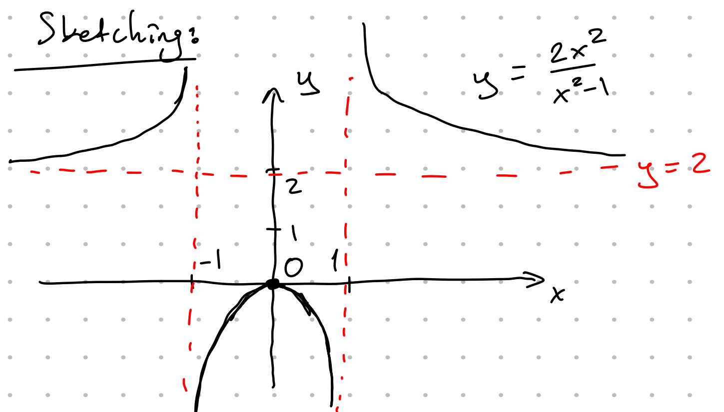
x		-1		0		1	
f''	$+$		$-$				$+$
f	conc. up		conc. down				conc. up

Conc. up on $(-\infty, -1) \cup (1, \infty)$

conc. down on $(-1, 1)$

Even though concavity changes at -1 and 1, they are not in the domain so they are not inflection points.

Sketching:



Example Sketch $f(x) = \frac{x^2}{\sqrt{x+1}}$

A. Since we have $\sqrt{x+1}$ in the denom, $x + 1 > 0$

Domain $= (-1, \infty)$

B. y-int: $f(0) = \frac{0}{\sqrt{1}} = 0$

x-int: $f(x) = 0$ $\quad \frac{x^2}{\sqrt{x+1}} = 0 \iff x = 0$

C. No symmetries.

D. $\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{x^2}{\sqrt{x+1}} = \infty$ no horizontal asymp.

$\lim_{x \to -1^+} \frac{x^2 \to +1}{\sqrt{x+1} \to 0^+} = +\infty$ vertical asymp at $x = -1$.

E. $f(x) = \frac{x^2}{(x+1)^{1/2}}$ $\quad f'(x) = \frac{2x(x+1)^{1/2} - x^2 \frac{1}{2}(x+1)^{-1/2}}{x+1}$

$$f'(x) = \frac{\frac{1}{2}(x+1)^{-1/2} \cdot x \left[4(x+1) - x\right]}{x+1} = \frac{x\left[4x+4-x\right]}{2(x+1)^{1/2}(x+1)}$$

$$= \frac{x\left[3x+4\right]}{2(x+1)^{3/2}} = \frac{3x^2+4x}{2(x+1)^{3/2}}$$

$(x+1)^{3/2}$ is always positive in $(-1,\infty)$

$f'(x) = 0 \quad x(3x+4) = 0 \Rightarrow x = 0, -\frac{4}{3}$

$x=0$ is the only critical point ($-\frac{4}{3}$ is not in the domain)

x	-1		0	
$f'(x)$		$-$		$+$
$f(x)$		dec ↘		inc ↗

inc on $(0,\infty)$
dec on $(-1,0)$

F. local min at $x=0$

G. $f'(x) = \frac{3x^2+4x}{2(x+1)^{3/2}} \qquad f''(x) = \frac{(6x+4)(2(x+1)^{3/2}) - (3x^2+4x)\,3(x+1)^{1/2}}{4(x+1)^3}$

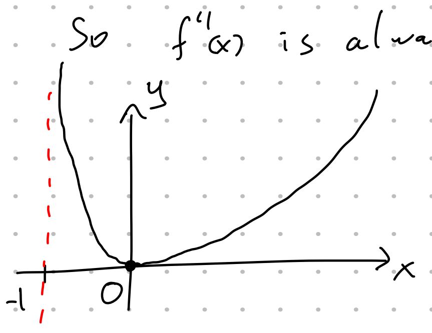
$$f''(x) = \frac{\cancel{(x+1)^{1/2}}\left[4(3x+2)(x+1) - 3(3x^2+4x)\right]}{4(x+1)^{\cancel{3}\,5/2}}$$

$$= \frac{(12x+8)(x+1) - 9x^2 - 12x}{4(x+1)^{5/2}} = \frac{\overset{3}{\cancel{12}}x^2 + 8x + \cancel{12x} + 8 - \cancel{9x^2} - \cancel{12x}}{4(x+1)^{5/2}}$$

$$= \frac{3x^2+8x+8}{4(x+1)^{5/2}} \longrightarrow b^2 - 4ac = 64 - 4(3 \cdot 8) < 0$$

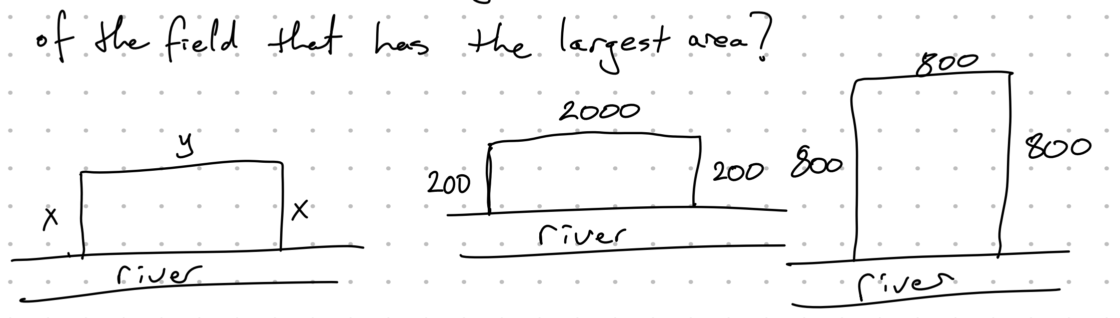
$3 \cdot 8 = 24$ — no real roots

So $f''(x)$ is always positive! f is always conc. up.



4.7 Optimization Problems

Example A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area?



$2x+y = 2400 \longrightarrow y = 2400 - 2x$

$A = xy \qquad A(x) = x(2400-2x) = 2400x - 2x^2$

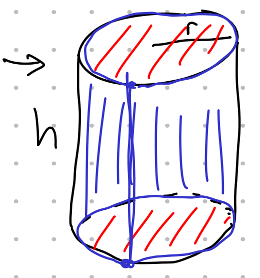
$A'(x) = 2400 - 4x = 0 \Rightarrow x = \frac{2400}{4} = 600$

x	0		600		1200
A'		$+$		$-$	
A		↗		↘	

$x = 600$ is the global max

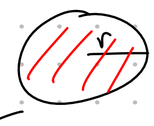
$y = 2400 - 2x = 2400 - 1200 = 1200$

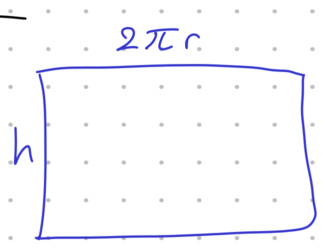
Example A cylindrical can is to be made to hold 1 L of oil. Find the dimensions that will minimize the cost of the metal to manufacture the can.



$V = \pi r^2 h = 1\,L$

2 circles





$A = 2\pi r^2 + 2\pi r h$

$h = \frac{1}{\pi r^2} \qquad A(r) = 2\pi r^2 + 2\cancel{\pi}\cancel{r}\left(\frac{1}{\cancel{\pi} r^{\cancel{2}}}\right)$

$= 2\pi r^2 + \frac{2}{r} = 2\pi r^2 + 2r^{-1}$

$A' = 4\pi r - 2r^{-2}$

$A' = 4\pi r - \frac{2}{r^2} = \frac{4\pi r^3 - 2}{r^2} = 0 \qquad 4\pi r^3 - 2 = 0$

$r^3 = \frac{2}{4\pi} = \frac{1}{2\pi}$

$r = \sqrt[3]{\frac{1}{2\pi}}$

r		$\sqrt[3]{\frac{1}{2\pi}}$	
A'	$-$		$+$
A	dec ↘		inc ↗

$r = \sqrt[3]{\frac{1}{2\pi}}$ is the global min.

$h = \frac{1}{\pi r^2} = \frac{1}{\pi\left(\frac{1}{2\pi}\right)^{2/3}}$