sphere $V = \frac{4}{3}\pi r^3$ cylinder $V = \pi r^2 h$ rectangular box $V = dwh$

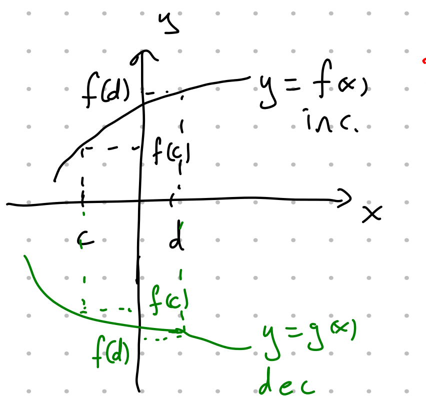
4.3 How derivatives affect the shape of a graph

Q) What does f' say about f?

A) <u>Increasing / Decreasing Test</u>

a) If $f'(x) > 0$ on (a, b) then $f(x)$ is increasing on (a,b)

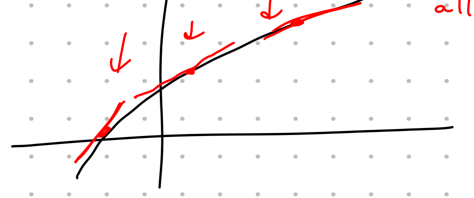
b) If $f'(x) < 0$ on (a, b) then $f(x)$ is decreasing on (a,b)



Increasing means if $c<d$ then $f(c)<f(d)$

Decreasing means if $c<d$ then $f(c)>f(d)$

slopes of the tangents are all positive > 0



<u>Example</u> Find where the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is inc. and where it is dec.

$f'(x) = 12x^3 - 12x^2 - 24x = 12x(x^2 - x - 2) = 12x(x-2)(x+1)$

Set $f'(x) = 0 \Rightarrow x = 0, -1, 2$

x		-1		0		2	
$f'(x)$	$-$		$+$		$-$		$+$
$f(x)$	dec		inc		dec		inc

$f'(1) = 12 \cdot 1 \cdot (1-2)(1+1)$

f is inc. on $(-1, 0) \cup (2, \infty)$

f is dec. on $(-\infty, -1) \cup (0, 2)$

<u>Local Extreme Values</u>

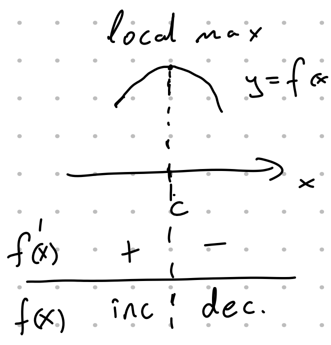
<u>The First Derivative Test</u>

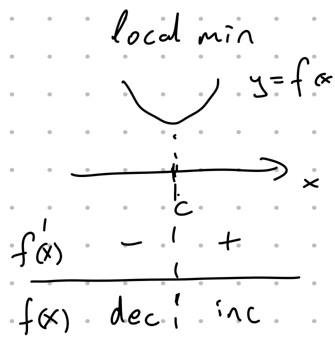
Suppose that c is a critical number of f.

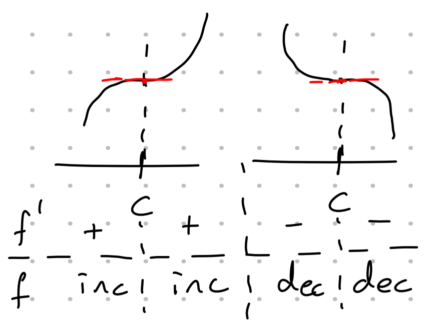
a) If f' changes from positive to negative at c, then f has a local max at c.

b) If f' changes from negative to positive at c, then f has a local min at c.

c) If the sign of f' does not change, then c is neither a local max nor min.







f inc $\Leftrightarrow f' > 0$

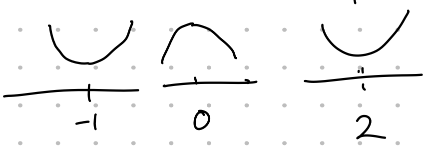
f dec $\Leftrightarrow f' < 0$

<u>Example</u> Find the local min and max values of $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$

x		-1		0		2	
$f'(x)$	$-$		$+$		$-$		$+$
$f(x)$	dec		inc		dec		inc

-1 and 2 are local mins

0 is a local max

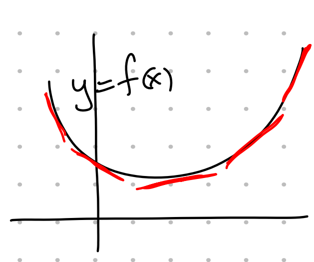


$f(-1) = 3(-1)^4 - 4(-1)^3 - 12(-1)^2 + 5 = 3 + 4 - 12 + 5 = 0$

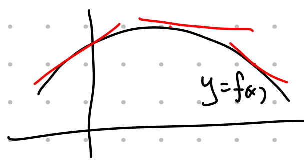
$f(2) = 3(2)^4 - 4(2)^3 - 12(2)^2 + 5 = -27$ → local min values

$f(0) = 5$ → local max value.

Q) What does f'' say about f?



the graph of $f(x)$ lies above any one of its tangent lines : f is <u>concave upward</u>



the graph of $f(x)$ lies below any one of its tangent lines : f is <u>concave downward</u>

<u>Concavity Test</u>

a) If $f''(x) > 0$, then f is concave up

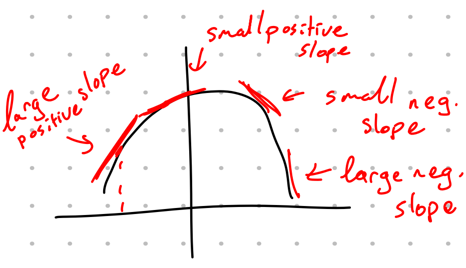
b) If $f''(x) < 0$, then f is concave down.

(Justification using inc/dec test)

Apply inc/dec test to f' (not f)

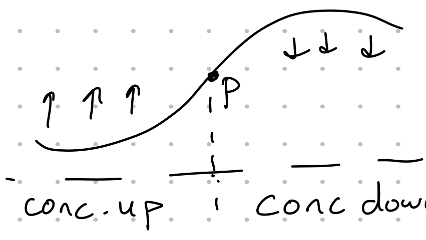
$f''(x) > 0 \Rightarrow f'$ is inc $\Rightarrow$ slopes of tangents are inc.

$f''(x) < 0 \Rightarrow f'$ is dec $\Rightarrow$ slopes of tangents are dec.



(slopes of the tangents are dec) conc. down

Definition A point P on $y=f(x)$ is called an inflection point if f is continuous there and concavity of f changes.



so P is an inflection point.

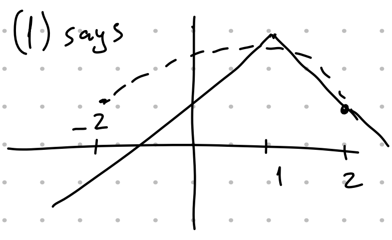
Example Sketch a possible graph of a function f that satisfies

1) $f'(x) > 0$ on $(-\infty, 1)$, $f'(x) < 0$ on $(1, \infty)$

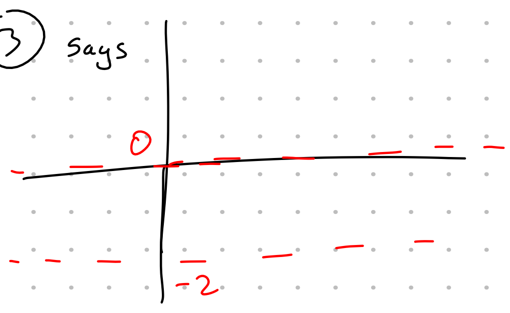
2) $f''(x) > 0$ on $(-\infty, -2) \cup (2, \infty)$, $f''(x) < 0$ on $(-2, 2)$

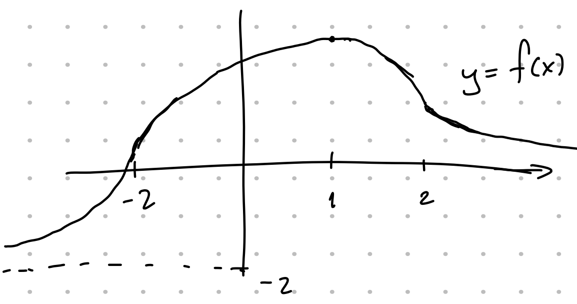
3) $\lim_{x \to -\infty} f(x) = -2$, $\lim_{x \to \infty} f(x) = 0$

(1) says



(3) says



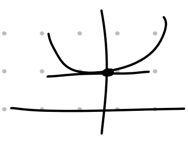


The Second Derivative Test

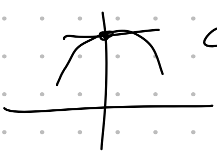
Suppose f'' is cont near c.

(a) If $f'(c) = 0$ and $f''(c) > 0$, then c is a local min.

(b) If $f'(c) = 0$ and $f''(c) < 0$, then c is a local max.



conc up $\Leftrightarrow f'' > 0$
$\Rightarrow$ local min



conc down $\Leftrightarrow f'' < 0$
$\Rightarrow$ local max

Remark: if $f'(c) = 0$ and $f''(c) = 0$ the 2nd der. test is inconclusive

Example $y = x^4 - 4x^3 = f(x)$

$f'(x) = 4x^3 - 12x^2 = 4x^2(x-3)$ $\quad x = 0, 3$ are critical numbers

$f''(x) = 12x^2 - 24x = 12x(x-2)$

$f'(0) = 0$ and $f''(0) = 0$ so the second der. test is inconclusive

$f'(3) = 0$ and $f''(3) = 12(3)(3-2) > 0 \Rightarrow 3$ is a local min.

At $x = 0, 2$ $\quad f''(x) = 0$. Those points might be inflection point

x		0		2		3	
f'	−		−		−		+
f''	+		−		+		+
f	conc up		conc down		conc up		conc up
f	dec		dec		dec		inc

changes changes $\Rightarrow$ 0, 2 are inflection points

local min at $x=3$

neither local max nor min at $x=0$ (by the first der. test)

4.4 Indeterminate Forms and L'Hospital's Rule

Q) How to compute $\lim_{x \to 1} \frac{\ln x}{x-1}$ $\begin{matrix} \to 0 \\ \to 0 \end{matrix}$ $\quad \frac{0}{0}$

Similarly, $\lim_{x \to \infty} \frac{\ln x}{x-1}$ $\begin{matrix} \to \infty \\ \to \infty \end{matrix}$ $\quad \frac{\infty}{\infty}$

A) We apply L'H rules: (We take derivative of top and bottom)

$$\lim_{x \to 1} \frac{\ln x}{x-1} = \lim_{x \to 1} \frac{(\ln x)'}{(x-1)'} = \lim_{x \to 1} \frac{\frac{1}{x}}{1} = 1$$

$$\lim_{x \to \infty} \frac{\ln x}{x-1} = \lim_{x \to \infty} \frac{(\ln x)'}{(x-1)'} = \lim_{x \to \infty} \frac{\frac{1}{x}}{1} = 0$$