Example (Related Rates)

If a snowball melts so that its surface area decreases at a rate of $1\,cm^2/min$, find the rate at which the diameter decreases when the diameter is $10\,cm$.

$\cancel{V = \frac{4}{3}\pi r^3}$ $A = 4\pi r^2$ surface Area

$A = A(t)$, $r = r(t)$

$-1 = A'$ $\frac{d}{dt}$ $d' ?$ when $d = 10$

$d = 2r$

$-1 = A' = 4\pi(2r\,r')$

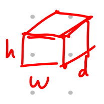
$d = 10 \Rightarrow r = 5$

$-1 = 8\pi(5\,r')$

$r' = \frac{-1}{40\pi}$ since $d = 2r$, $d' = 2r'$ $= \frac{-2}{40\pi} = -\frac{1}{20\pi}$

$\boxed{\frac{d\,(r(t))^2}{dt} = 2\,r(t)\cdot\frac{dr(t)}{dt} = 2r\cdot r'}$

We expect you to know the volume of a sphere $V = \frac{4}{3}\pi r^3$, the volume of a rectangular box



$V = hwd$, the volume of a circular cylinder

$V = \pi r^2 h$

Last time | **Example** Find the linearization of the function $f(x) = \sqrt{x+3}$ at $a = 1$ and use it to approximate the numbers $\sqrt{3.98}$ and $\sqrt{4.05}$. Are these approximations over-estimates or underestimates?

$L(x) = f(a) + f'(a)(x-a)$ $f(a) = \sqrt{1+3} = 2$

$f'(x) = \frac{1}{2}(x+3)^{-1/2}$ $f'(a) = \frac{1}{2}(1+3)^{-1/2} = \frac{1}{4}$

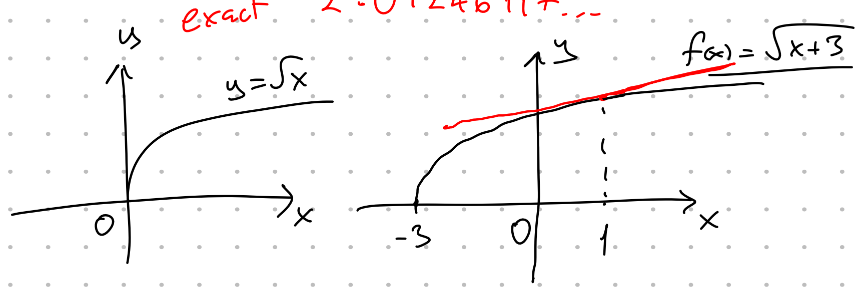
$L(x) = 2 + \frac{1}{4}(x-1)$

$\sqrt{3.98} = f(0.98) \approx L(0.98) = 2 + \frac{1}{4}(0.98-1) = 2 - 0.005 = 1.995$

exact $= 1.99499373\ldots$

$\sqrt{4.05} = f(1.05) \approx L(1.05) = 2 + \frac{1}{4}(1.05-1) = 2.0125$

exact $= 2.01246117\ldots$



The tangent line is always above the function $f(x) = \sqrt{x+3}$

$L(x) \geq f(x)$

Therefore, the approximations are over-estimates.

Example Given $f(2) = -1$, $f'(2) = 8$, $f(4) = 4$, and $f'(4) = -3$ find the linearization of $g(x) = x\,f(x^2)$ at $x = 2$.

$L(x) = g(a) + g'(a)(x-a) = g(2) + g'(2)(x-2)$

$g(2) = 2\,f(2^2) = 2\,f(4) = 8$

$g'(x) = (x)'\,f(x^2) + x\,(f(x^2))' = f(x^2) + x\,f'(x^2)\cdot 2x$

$g'(2) = f(4) + 8\,f'(4) = 4 - 24 = -20$

$L(x) = 8 - 20(x-2)$

Example Express the linearization of $f'(x)$ at $x = a$

$\boxed{L(x) = g(a) + g'(a)(x-a)}$ here $g(x) = f'(x)$

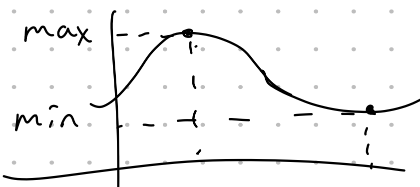
$L(x) = f'(a) + f''(a)(x-a) \approx f'(x)$ if x is near a.

4.1 Maximum and Minimum Values

Q| Minimum cost?

What is the shape of a can that maximizes volume?

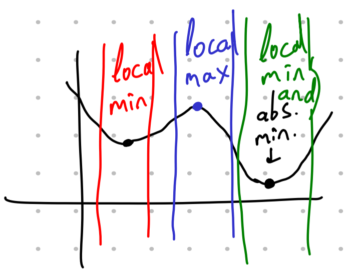
How to maximize profit?



Def^n Let c be a number in the domain D of a function f. Then $f(c)$ is the absolute maximum value of f on D if $f(c) \geq f(x)$ for all x in D.

$f(c)$ is the absolute minimum value of f on D if $f(c) \leq f(x)$ for all x in D.

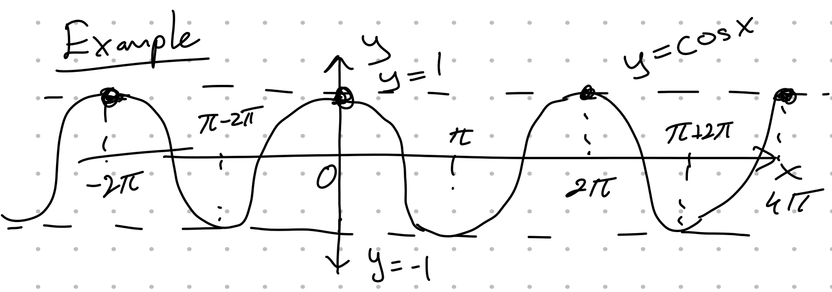
absolute max/min = global max/min



Def^n The number $f(c)$ is a

- local max. value of f if $f(c) \geq f(x)$ when x is near c
- local min. value of f if $f(c) \leq f(x)$ when x is near c

Example



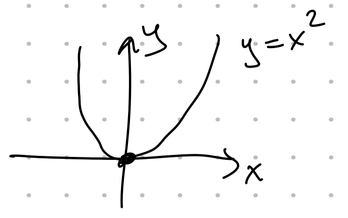
Global max value = 1 occurs infinitely many times when $\cos x = 1$.

$\cos x = 1$ for $x = 2\pi k$ for any integer k.

Global min = −1 occurs ∞'ly many times when $\cos x = -1$

$\cos x = -1$ at $x = \pi + 2\pi k$ for any integer k.

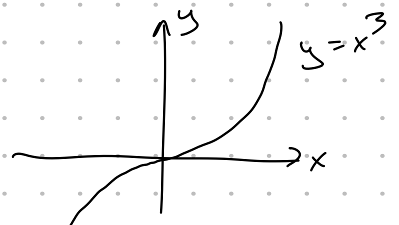
Example If $f(x) = x^2$ then $f(x) \geq 0$ and $f(0) = 0$



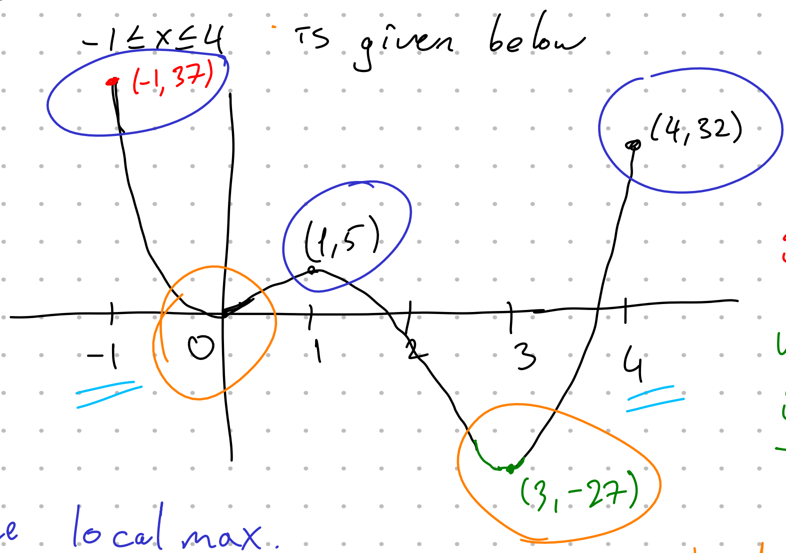
$f(0) = 0$ is a global min (unique)

But there is no global max.

Example If $f(x) = x^3$ then f has no global max or min.



Example The graph of the function $f(x) = 3x^4 - 16x^3 + 18x^2$ $-1 \leq x \leq 4$ is given below



We have a unique global max of 37 at $x = -1$.

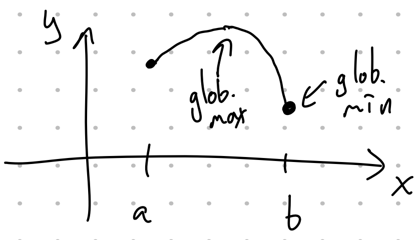
We have a unique global min of −27 at $x = 3$

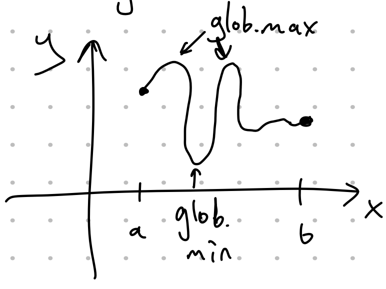
Three local max. at $x = 1$, $x = 4$ and $x = -1$

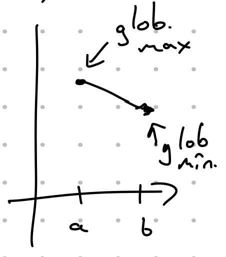
Two local min.s at $x = 0$ and $x = 3$

Thm (The Extreme Value Theorem)

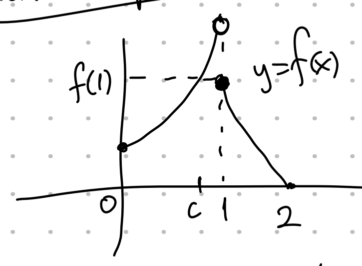
If f is continuous on a closed interval $[a, b]$ then f attains a global max and a global min on $[a, b]$





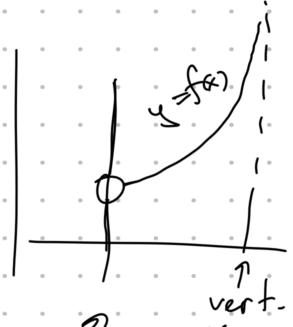


non-examples



does not have a global max!

f is not cont.



no glob. max or glob. min.

f is not defined on a closed interval