$y = n^x + x^n$ ← addition

$\log(A+B)$

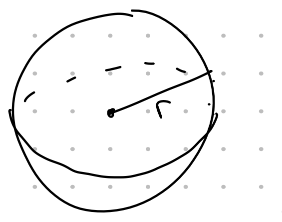
$\frac{dy}{dx} = y' = (n^x)' + (x^n)' = n^x \cdot \ln n + n\,x^{n-1}$

$(n^x)'$: exponential function — variable in exponent

$(x^n)'$: power function — power of the variable

3.9 Related Rates

Example Air is being pumped into a spherical balloon so that its volume increases at a rate of $100\ \text{cm}^3/\text{s}$. How fast is the radius increasing when the diameter is 50 cm?



$V = V(t)$: volume at time t (seconds)

$r = r(t)$: radius at time t (seconds)

$V' = 100\ \text{cm}^3/\text{s}$

when diameter $= 2r = 50$, $r' = ?$

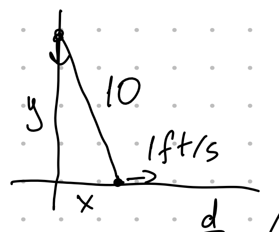
$\boxed{V = \frac{4}{3}\pi r^3}$ ← both sides are functions of t.

$\Rightarrow V' = \frac{4}{\cancel{3}}\pi \cancel{3} r^2 \cdot r'$ by the chain rule.

Since $2r = 50$, $r = 25$

$100 = 4\pi(25)^2 \cdot r' \Rightarrow r' = \frac{100}{4\pi(25)^2} = \frac{1}{25\pi}\ \text{cm/s}$

Example A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft/s, how fast is the top of the ladder sliding down the wall when the bottom is 6 ft from the wall?



Note that x and y are functions of time t.

$x^2 + y^2 = 10^2$

$x' = 1$

when $x = 6$, $y' = ?$

$\frac{d}{dt}$

$2x\,x' + 2y\,y' = 0$ or $x\,x' + y\,y' = 0$

$6(1) + y\,y' = 0$

$y' = \frac{-6}{y} = -\frac{6}{8} = -\frac{3}{4}$

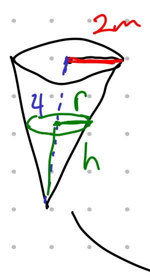
($y = 8$ when $x = 6$ since $x^2 + y^2 = 10^2$)

Sliding down at a rate of $\frac{3}{4}$ ft/s

(negative in front of $\frac{3}{4}$ represents the fact that y is decreasing)

Example

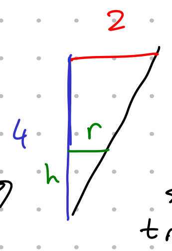
A water tank has the shape of an inverted circular cone with base radius 2 m and height 4 m. If water is being pumped into the tank at a rate of $2\ \text{m}^3/\text{min}$, find the rate at which the water level is rising when the water is 3 m deep.



$V' = 2\ \text{m}^3/\text{min}$

When $h = 3$, $h' = ?$

① $V = \frac{1}{3}\pi r^2 h$



using similar triangles:

② $\frac{r}{2} = \frac{h}{4}$

$h = 2r$ subs. in ①

this could have been shorter if we used $r = \frac{h}{2}$ (instead of $h = 2r$)

$V = \frac{1}{3}\pi r^2 (2r) = \frac{2\pi}{3} r^3$

when $h = 3$

$3 = 2r$

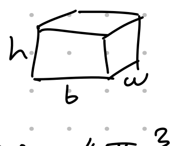
$r = \frac{3}{2}$

$V' = \frac{2\pi}{3}(3r^2) r' = 2\pi r^2 r'$

$\cancel{2} = \cancel{2}\pi\left(\frac{3}{2}\right)^2 r' \Rightarrow r' = \left(\frac{2}{3}\right)^2 \cdot \frac{1}{\pi}$

$h' = 2r' \longrightarrow h' = 2r' = 2\left(\frac{2}{3}\right)^2 \cdot \frac{1}{\pi} = \frac{8}{9\pi}\ \frac{\text{m}}{\text{min}}$

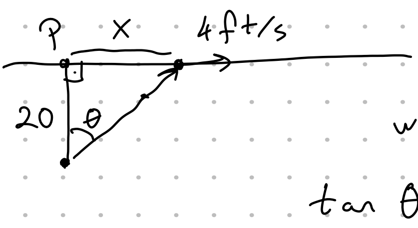
$V = \pi r^2 h$



$V = hbw$

$V = \frac{4}{3}\pi r^3$

Example A man walks along a straight path at a speed of 4 ft/s. A searchlight is located on the ground 20 ft from the path and is kept focused on the man. At what rate is the searchlight rotating when the man is 15 ft from the point on the path closest to the searchlight?



$x' = 4$

when $x = 15$, $\theta' = ?$

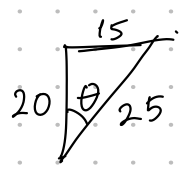
$\tan\theta = \frac{x}{20}$

(everything is a function of t)

$\frac{d}{dt}$

$$\sec^2(\theta)\cdot\theta' = \frac{x'}{20} = \frac{4}{20} = \frac{1}{5}$$

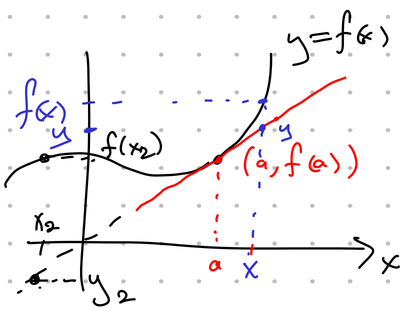
what is $\sec\theta$ when $x=15$?



$$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{\frac{20}{25}} = \frac{25}{20} = \frac{5}{4}$$

$$\sec^2(\theta)\,\theta' = \left(\frac{5}{4}\right)^2\theta' = \frac{1}{5} \qquad \theta' = \left(\frac{4}{5}\right)^2\cdot\frac{1}{5} = \frac{16}{125}$$

3.10 Linear Approximations



$y - f(a) = f'(a)(x-a)$

or $y = f(a) + f'(a)(x-a)$

approximately equal to

$f(x) \approx f(a) + f'(a)(x-a)$

when x is near a.

$L(x) = f(a) + f'(a)(x-a)$ is called the linear approximation or tangent line approximation or linearization of f.

Example Find the linearization of $f(x) = \sqrt{x+3}$ at $a=1$ $(x=1)$ and use it to approximate $\sqrt{3.98}$ and $\sqrt{4.05}$. Are these approximations over-estimates or under-estimates?

$f(a) = f(1) = \sqrt{1+3} = \sqrt{4} = 2$

$f'(x) = \frac{1}{2}(x+3)^{-1/2} \qquad f'(a) = f'(1) = \frac{1}{2}(1+3)^{-1/2} = \frac{1}{2}\frac{1}{\sqrt{4}} = \frac{1}{4}$

$f(x) \approx L(x) = f(a) + f'(a)(x-a) = 2 + \frac{1}{4}(x-1)$

$\sqrt{3.98} = \sqrt{0.98+3} = f(0.98) \approx L(0.98) = 2 + \frac{1}{4}(0.98-1)$

$= 2 - \frac{0.02}{4} = 2 - 0.005$

$= 1.995$

Exact value: 1.99499373

$\sqrt{4.05} = f(1.05) \approx L(1.05) = 2 + \frac{1}{4}(0.05) = 2.0125$

$2.01246\ldots$