Logarithmic Differentiation

Example Find y' if $y = \dfrac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}$

To simplify this problem, we start by taking $\ln$ of both sides

$$\ln y = \ln\left(\frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}\right)$$

$\ln(AB) = \ln A + \ln B$

$\ln\left(\frac{A}{B}\right) = \ln A - \ln B$

$\ln(A^k) = k\ln A$

$$\ln y = \ln\left(x^{3/4}\sqrt{x^2+1}\right) - \ln\left((3x+2)^5\right)$$

$\sqrt{x^2+1} = (x^2+1)^{1/2}$

$$\ln y = \frac{3}{4}\ln x + \frac{1}{2}\ln(x^2+1) - 5\ln(3x+2)$$

$\frac{d}{dx}$

$$\frac{y'}{y} = \frac{3}{4}\cdot\frac{1}{x} + \frac{1}{\cancel{2}}\,\frac{\cancel{2}x}{x^2+1} - 5\,\frac{3}{3x+2}$$

$(\ln(g(x)))' = \dfrac{g'(x)}{g(x)}$

$$y' = y\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right)$$

$$= \left(\frac{x^{3/4}\sqrt{x^2+1}}{(3x+2)^5}\right)\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right)$$

We know how to differentiate $(g(x))^k$ and $k^{g(x)}$.

What about $(f(x))^{(g(x))}$?

Example Differentiate $y = x^{\sqrt{x}}$ We don't know any direct method for this kind of function!

Solution 1

$\frac{d}{dx}$ of both sides

$$\ln y = \ln x^{\sqrt{x}} = \sqrt{x}\,\ln x = x^{1/2}\ln x$$

$$\frac{y'}{y} = \frac{1}{2}x^{-1/2}\ln x + x^{1/2}\cdot\frac{1}{x} = \frac{\ln x}{2\sqrt{x}} + \frac{1}{\sqrt{x}} = \frac{\ln x + 2}{2\sqrt{x}}$$

$$\Rightarrow\quad y' = y\left(\frac{\ln x + 2}{2\sqrt{x}}\right) = x^{\sqrt{x}}\left(\frac{\ln x + 2}{2\sqrt{x}}\right)$$

Solution 2

$$y = x^{\sqrt{x}} = e^{\ln(x^{\sqrt{x}})}$$

$e^{\ln A} = A$

$$y = e^{\sqrt{x}\ln x}$$

$$y' = \underbrace{e^{\sqrt{x}\ln x}}_{y}\left(\sqrt{x}\ln x\right)' = x^{\sqrt{x}}\left(\sqrt{x}\ln x\right)' = x^{\sqrt{x}}\left(\frac{\ln x + 2}{2\sqrt{x}}\right)$$

The number e as a limit

$\boxed{\ln 1 = 0}$

$f(x) = \ln x$ then $f'(x) = \frac{1}{x}$ so $f'(1) = 1$

$$f'(1) = \lim_{h\to 0}\frac{f(1+h) - f(1)}{h} = \lim_{x\to 0}\frac{f(1+x) - f(1)}{x}$$

$$f'(1) = \lim_{x\to 0}\frac{\ln(1+x) - \ln(1)}{x} = \lim_{x\to 0}\frac{1}{x}\ln(1+x)$$

$$1 = f'(1) = \lim_{x\to 0}\ln(1+x)^{1/x}$$

$$\Downarrow$$

$$e^1 = e = e^{\left(\lim_{x\to 0}\ln(1+x)^{1/x}\right)} = \lim_{x\to 0}\cancel{e}^{\cancel{\ln}(1+x)^{1/x}}$$

$$\Rightarrow\quad e = \lim_{x\to 0}(1+x)^{1/x}$$

Set $n = \frac{1}{x}$

$n \to \infty$ as $x \to 0^+$

$$\Rightarrow\quad e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$$

for example $e \approx \left(1 + \frac{1}{1000}\right)^{1000}$

Recall $f(x) = \ln|x|$ then $f'(x) = \frac{1}{x}$ (from last time)

Section 3.7

Particles moving on a straight line

Example The position of a particle is given by the equation $s = f(t) = t^3 - 6t^2 + 9t$ where t is measured in seconds and s in meters.

a) Find the velocity at time t.

$$v = s' = 3t^2 - 12t + 9$$

b) What is the velocity after 2 s? After 4 s?

$$v(2) = 3(2)^2 - 12(2) + 9 = -3 \text{ m/s}$$

$$v(4) = 3(4)^2 - 12(4) + 9 = 9 \text{ m/s}$$

c) When is the particle at rest?

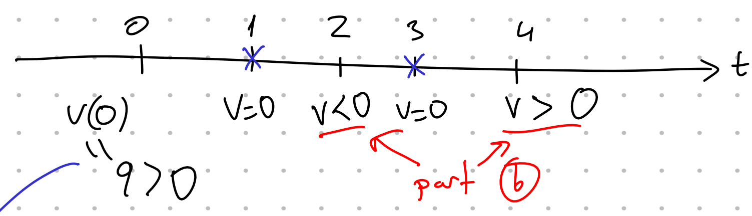
Set $v = 0$ $\qquad 3t^2 - 12t + 9 = 0$

$$t^2 - 4t + 3 = 0$$

$$(t-3)(t-1) = 0 \Rightarrow t = 1, 3$$

d) When is the particle moving forward (in the positive direction)?

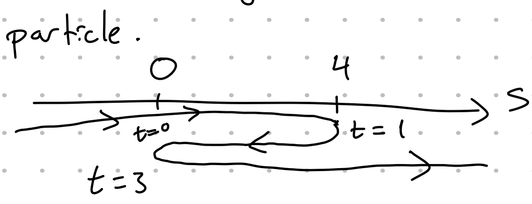
from ⓒ we see that at $t=1,3$ the particle is at rest.



t	$(-\infty,1)$	$(1,3)$	$(3,\infty)$
v	$+$	$-$	$+$

so the particle is moving forward on $(-\infty,1)\cup(3,\infty)$

e) Draw a diagram to represent the motion of the particle.



$s(1) = 1^3 - 6(1)^2 + 9(1)$
$= 1-6+9 = 4\,m$
$s(3) = 3^3 - 6(3)^2 + 9(3)$
$= 0\ m$

f) Find the total distance traveled during the first five seconds.

Not the same thing as $s(5)-s(0)$ or $|s(5)-s(0)|$!

We need to compute

$|s(1)-s(0)| + |s(3)-s(1)| + |s(5)-s(3)|$ since at $t=1$ and $t=3$ we change direction.

$s(0)=0,\quad s(1)=4,\quad s(3)=0,\quad s(5)=5^3-6(5)^2+9(5)$
$= 20\,m$

$|4-0| + |0-4| + |20-0|$
$= 28\ m$

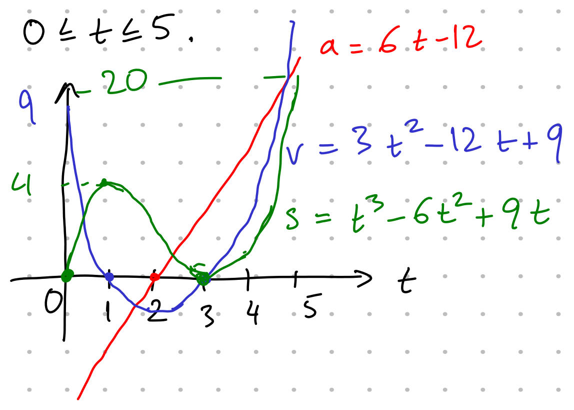
(Total displacement during the first 5 seconds $= s(5)-s(0)$)
$\neq$ Total distance traveled.

g) Find the acceleration at time t and after 4 s.

$a = v' = (3t^2-12t+9)' = 6t-12\ m/s^2$

$a(4) = 24-12 = 12\ m/s^2$

h) Graph the position, velocity, and acceleration functions for $0 \le t \le 5$.



i) When is the particle speeding up? When is it slowing down?

Recall speed $= |\text{velocity}|$

Note that $a<0$ for $t<2$
$a>0$ for $t>2$

similarly $v<0$ for $1<t<3$
$v>0$ for $t<1$ or $3<t$ (part d)

t		1		2		3	
v	$+$		$-$		$-$		$+$
a	$-$		$-$		$+$		$+$
	slowing down		speeding up		slowing down		speeding up

slowing down on $(0,1)\cup(2,3)$

Speeding up on $(1,2)\cup(3,5)$