(1 point) Library/UVA-Stew5e/setUVA-Stew5e-C03S03-RatesofChange/3-3-08.pg

If a ball is thrown vertically upward from the roof of a 64 foot tall building with a velocity of 64 ft/sec, its height in feet after t seconds is $s(t) = 64 + 64t - 16t^2$. What is the maximum height the ball reaches? $s(2)$

What is the velocity of the ball when it hits the ground (height 0)?

$s(t) = 64 + 64t - 16t^2$ $\qquad s(0) = 64$

Set $v = 0$ $\qquad v = s' = 64 - 32t = 0$ $\qquad t = 2$

a) $s(2) = 64 + 64(2) - 16(4)$

b) Set $s = 0$ $\qquad 64 + 64t - 16t^2 = 0$ $\qquad t^2 - 4t - 4 = 0$

$(t-2)^2 - 8 = 0$ $\qquad (t-2)^2 = 8$ $\qquad t - 2 = \pm\sqrt{8}$

$t = 2 \pm \sqrt{8}$ $\qquad t = 2 + \sqrt{8}$

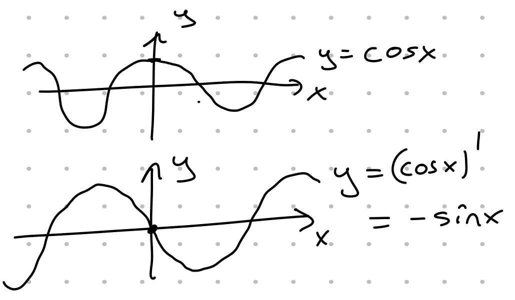
$v(2+\sqrt{8}) = 64 - 32(2+\sqrt{8})$ | speed $= |$velocity$|$

speed: always non-negative; velocity: can be negative

Last time: $(\sin x)' = \cos x$

$(uv)' = u'v + uv'$ $\qquad \left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$

$\frac{d}{dx}\cos x = -\sin x$



$\frac{d}{dx}\tan x = \left(\frac{\sin x}{\cos x}\right)' = \frac{(\sin x)' \cdot \cos x - \sin x (\cos x)'}{(\cos x)^2}$

$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$ $\qquad \left(\sec x = \frac{1}{\cos x}\right)$

$(\sin x)' = \cos x$ $\qquad (\tan x)' = \sec^2 x$ $\qquad (\sec x)' = \sec x \tan x$

$(\cos x)' = -\sin x$ $\qquad (\cot x)' = -\csc^2 x$ $\qquad (\csc x)' = -\csc x \cot x$

Example Differentiate $y = \frac{\sec x}{1 + \tan x}$. For what values of x does the graph have a horizontal tangent?

$y' = \frac{(\sec x)'(1+\tan x) - (\sec x)(1+\tan x)'}{(1+\tan x)^2}$

$\boxed{\begin{aligned} &\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} \\ &\tan^2 x + 1 = \sec^2 x \\ &\tan^2 x - \sec^2 x = -1 \end{aligned}}$

$= \frac{\sec x \tan x (1 + \tan x) - \sec x \sec^2 x}{(1+\tan x)^2}$

$= \frac{\sec x \tan x + \sec x \tan^2 x - \sec x \sec^2 x}{(1+\tan x)^2}$

$= \frac{\sec x \tan x + \sec x(\tan^2 x - \sec^2 x)}{(1+\tan x)^2} = \frac{\sec x \tan x - \sec x}{(1+\tan x)^2}$

$= \frac{\sec x(\tan x - 1)}{(1+\tan x)^2}$ $\qquad$ Set $y' = 0 = \frac{\sec x(\tan x - 1)}{(1+\tan x)^2}$

if and only if $\sec x(\tan x - 1) = 0$

iff $\sec x = 0$ $\qquad$ or $\qquad \tan x - 1 = 0$

$\left(\frac{1}{\cos x} = 0\right.$ never holds! $\qquad \tan x = 1$

$x = \frac{\pi}{4}, \frac{\pi}{4} + \pi, \frac{\pi}{4} + 2\pi, \ldots$

$\frac{\pi}{4} - \pi, \frac{\pi}{4} - 2\pi, \ldots$

Example Find the 27^{th} derivative of $\cos x$

$y^{(0)} = \cos x$ $\qquad y' = -\sin x$ $\qquad y'' = -\cos x$ $\qquad y''' = \sin x$

$y^{(4)} = \cos x$ $\qquad y^{(5)} = -\sin x$ $\qquad \ldots$

$y^{(8)} = \cos x$

$y^{(24)} = \cos x$ $\qquad y^{(25)}$ $\qquad y^{(26)}$ $\qquad y^{(27)} = \sin x$

$\lim_{x\to 0} \frac{\sin x}{x} = 1$ $\qquad \lim_{x\to 0} \frac{\cos x - 1}{x} = 0$

Example Find $\lim_{x\to 0} \frac{\sin 7x}{4x} = \lim_{x\to 0} \frac{7}{4}\frac{\sin 7x}{7x}$

set $y = 7x$ then $y \to 0$ iff $x \to 0$ $\qquad = \frac{7}{4}\lim_{y\to 0}\frac{\sin y}{y} = \frac{7}{4}\cdot 1 = \frac{7}{4}$

Example Find $\lim_{x\to 0} x\cot x = \lim_{x\to 0} x\frac{\cos x}{\sin x}$

$= \lim_{x\to 0} \frac{\cos x}{\frac{\sin x}{x}} = \frac{\lim_{x\to 0}\cos x}{\lim_{x\to 0}\frac{\sin x}{x}} = \frac{1}{1} = 1$

Example

$\lim_{x\to 2} \frac{\sin(x-2)}{x^2+2x-8} = \lim_{x\to 2}\frac{\sin(x-2)}{(x-2)(x+4)} = \lim_{x\to 2}\frac{1}{(x+4)}\cdot\frac{\sin(x-2)}{(x-2)}$

$= \lim_{x\to 2}\frac{1}{x+4}\cdot\lim_{x\to 2}\frac{\sin(x-2)}{(x-2)} = \frac{1}{6}\cdot 1 = \frac{1}{6}$

set $y = x - 2$ ($x \to 2$ iff $y \to 0$)

$\lim_{y\to 0}\frac{\sin y}{y}$

3.4 The Chain Rule

Q) How to compute $\frac{d}{dx}F(x)$ where $F(x)=\sqrt{x^2+1}$?

A) If $F=f\circ g$ i.e. $F(x)=f(g(x))$

$$F'(x)=f'(g(x))\cdot g'(x) \quad \text{"The Chain Rule"}$$

In Leibniz notation if $y=f(u)$ and $u=g(x)$

then $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$

Another notation $(f\circ g)'(x)=f'(g(x))\cdot g'(x)$

Example Find $F'(x)$ if $F(x)=\sqrt{x^2+1}$

$f(x)=\sqrt{x}=x^{1/2}$ $\quad g(x)=x^2+1$ $\quad$ Then $f(g(x))=\sqrt{x^2+1}=F(x)$

$f'(x)=\frac{1}{2}x^{-1/2}$ $\quad g'(x)=2x$

$$F'(x)=f'(g(x))\cdot g'(x)=\frac{1}{\cancel{2}}(x^2+1)^{-1/2}\cdot \cancel{2}x=\frac{x}{\sqrt{x^2+1}}$$

Example Differentiate a) $y=\sin(x^2)$
b) $w=\sin^2 x=(\sin x)^2$

a) $f(x)=\sin x$ $\quad g(x)=x^2$ $\quad y=f(g(x))=\sin x^2$
$f'(x)=\cos x$ $\quad g'(x)=2x$
$y'=f'(g(x))\cdot g'(x)=\underline{\cos(x^2)}\cdot\underline{2x}$

b) $f(x)=x^2$ $\quad g(x)=\sin x$ $\quad w=f(g(x))=(\sin x)^2$
$f'(x)=2x$ $\quad g'(x)=\cos x$
$w'=f'(g(x))\cdot g'(x)=2\sin x\cos x$

Example If $y=(x^3-1)^{100}$, find y'.

$f(x)=x^{100}$ $\quad g(x)=x^3-1$ $\quad y=f(g(x))$
$f'(x)=100x^{99}$ $\quad g'(x)=3x^2$
$y'=100(x^3-1)^{99}\cdot 3x^2$