<u>Last time</u> $(x^n)' = nx^{n-1}$, $(cf(x))' = cf'(x)$

and $(f(x) \pm g(x))' = f'(x) \pm g'(x)$

e.g. $(x^8 + 12x^5 - 4x\sqrt{x})' = 8x^7 + 12(5x^4) - 4(x\sqrt{x})' = 8x^7 + 60x^4 - 4(\frac{3}{2}x^{1/2})$

Note that $x\sqrt{x} = x \cdot x^{1/2} = x^{3/2}$. Thus, $(x\sqrt{x})' = \frac{3}{2}x^{1/2}$

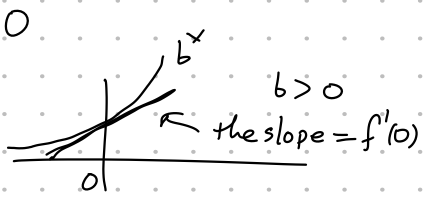
<u>Example</u> The equation of motion of a particle is

$s = 2t^3 - 5t^2 + 3t + 4$. Find the acceleration as a function of time. <u>Recall</u> $a = v' = \frac{dv}{dt}$ $\quad v = s' = \frac{ds}{dt}$

$v = s' = 2(3t^2) - 5(2t) + 3 + 0 = 6t^2 - 10t + 3$

$a = v' = 6(2t) - 10 = 12t - 10$

<u>Exponential Functions</u>



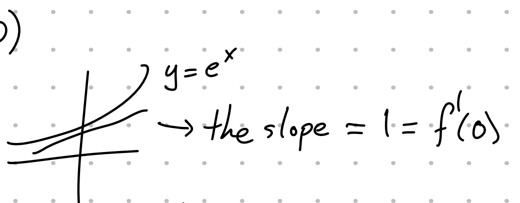
$f(x) = b^x$

$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{b^h - b^0}{h} = \lim_{h \to 0} \frac{b^h - 1}{h}$

$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{b^{x+h} - b^x}{h} = \lim_{h \to 0} \frac{b^x(b^h - 1)}{h}$

$(b^x)' = b^x \left(\lim_{h \to 0} \frac{b^h - 1}{h}\right) = b^x f'(0)$

Recall e is the number so that



So $\underline{(e^x)' = e^x f'(0) = e^x \cdot 1 = e^x}$ $\quad$ | In particular, $f'(0) = \lim_{h \to 0} \frac{e^h - 1}{h} = 1$ (if $f = e^x$)

<u>Example</u> Find f' and f'' if $f(x) = e^x - x$

$f'(x) = (e^x - x)' = (e^x)' - (x)' = e^x - 1$

$f''(x) = (e^x - 1)' = e^x$

<u>Example</u> At what <u>point</u> on the curve $\underline{y = e^x}$ is the tangent line parallel to the line $y = 2x$?

The slope of $y = 2x$ line is 2.

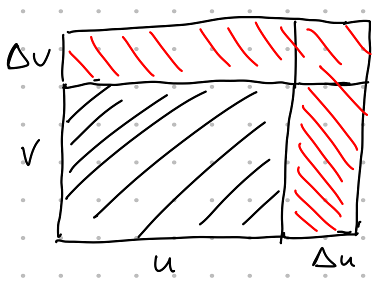
$y' = (e^x)' = e^x = 2 \Rightarrow x = \ln 2 \Rightarrow y = e^{\ln 2} = 2$

$(x, y) = (\ln 2, 2)$

3.2 The Product and Quotient Rules

<u>Q]</u> find $\frac{d}{dx}(f(x)g(x)) = (f(x)g(x))'$

<u>A]</u> Set $u = f(x)$ $\quad v = g(x)$. We want to find $(uv)'$.



Say x changed by Δx and that changes u by Δu and v by Δv

Red Area = change in the product $= \Delta(uv)$

$\Delta(uv)$ = Big Rectangle − Small Rectangle

$= (u + \Delta u)(v + \Delta v) - uv = \cancel{uv} + \underline{u\Delta v + v\Delta u + \Delta u \Delta v} - \cancel{uv}$

$\Delta(uv) = u\Delta v + v\Delta u + \Delta u \Delta v$

So $\underline{(uv)'} = \lim_{\Delta x \to 0} \frac{\Delta(uv)}{\Delta x}$ $\quad \leftarrow \left(h'(x) = \lim_{\Delta x \to 0} \frac{\Delta h}{\Delta x}\right)$

$= \lim_{\Delta x \to 0} u\frac{\Delta v}{\Delta x} + v\frac{\Delta u}{\Delta x} + \frac{\Delta u}{\Delta x}\underset{0}{\Delta v}$

$= \underline{uv' + vu'} + \cancel{u' \cdot 0}$ $\quad 0$

$(uv)' = u'v + uv'$ $\leftarrow$ The Product Rule

$\frac{d}{dx}(f(x)g(x)) = \frac{df(x)}{dx} \cdot g(x) + f(x)\frac{d}{dx}g(x)$

<u>Example</u> If $f(x) = xe^x$, find $f'(x), f''(x), f'''(x), \ldots$

$f'(x) = (x)'e^x + x(e^x)' = e^x + xe^x = (x+1)e^x$

$f''(x) = 1e^x + (x+1)e^x = (x+2)e^x$

$f'''(x) = (x+3)e^x$ $\quad$ so in general $f^{(n)}(x) = (x+n)e^x$

<u>Example</u> Find f' if $f(t) = \sqrt{t}(a + bt)$

<u>Sol#1</u> $f'(t) = (\sqrt{t})'(a + bt) + \sqrt{t}(a + bt)'$ $\quad \sqrt{t} = t^{1/2}$

$= \frac{1}{2}t^{-1/2}(a + bt) + \sqrt{t}(0 + b)$

<u>Sol#2</u> $f(t) = \sqrt{t}(a + bt) = a\sqrt{t} + bt\sqrt{t} = at^{1/2} + bt^{3/2}$

$f'(t) = a(\frac{1}{2}t^{-1/2}) + b(\frac{3}{2}t^{1/2})$

<u>Example</u> If $f(x) = \sqrt{x}\,g(x)$, where $g(4) = 2$ and $g'(4) = 3$, find $f'(4)$.

$\sqrt{x} = x^{1/2}$

$f'(x) = (\sqrt{x})'g(x) + \sqrt{x}\,g'(x) = \frac{1}{2}x^{-1/2}g(x) + \sqrt{x}\,g'(x)$

$f'(4) = \frac{1}{2}4^{-1/2}g(4) + \sqrt{4}\,g'(4) = \frac{1}{\cancel{2}} \cdot \frac{1}{\sqrt{4}}(\cancel{2}) + \sqrt{4}(3)$

$= \frac{1}{2} + 6 = \frac{13}{2}$

The Quotient Rule

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{\frac{d}{dx}f(x)\cdot g(x) - f(x)\cdot \frac{d}{dx}g(x)}{(g(x))^2}$$

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

Example let $y = \frac{x^2+x-2}{x^3+6}$. Find y'.

$$y' = \frac{(x^2+x-2)'(x^3+6) - (x^2+x-2)(x^3+6)'}{(x^3+6)^2}$$

$$y' = \frac{(2x+1)(x^3+6) - (x^2+x-2)(3x^2)}{(x^3+6)^2}$$

$$= \frac{2x^4 + 12x + x^3 + 6 - 3x^4 - 3x^3 + 6x^2}{(x^3+6)^2}$$

$$= \frac{-x^4 - 2x^3 + 6x^2 + 12x + 6}{x^6 + 12x^3 + 36}$$

Example Find an equation of the tangent line to $y = \frac{e^x}{1+x^2}$ at $\left(1, \frac{e}{2}\right)$

the slope $= m = y'\big|_{x=1}$ $\qquad y' = \frac{(e^x)'(1+x^2) - (1+x^2)'\cdot e^x}{(1+x^2)^2}$

$$y' = \frac{e^x(1+x^2) - 2x e^x}{(1+x^2)^2} \qquad y'\big|_{x=1} = \frac{e(1+1) - 2e}{(1+1)^2} = 0$$

Thus, $\quad y - \frac{e}{2} = m(x-1) = 0(x-1) = 0$

$$y = \frac{e}{2}$$

Example Find F' if $F(x) = \frac{3x^2 + 2\sqrt{x}}{x}$

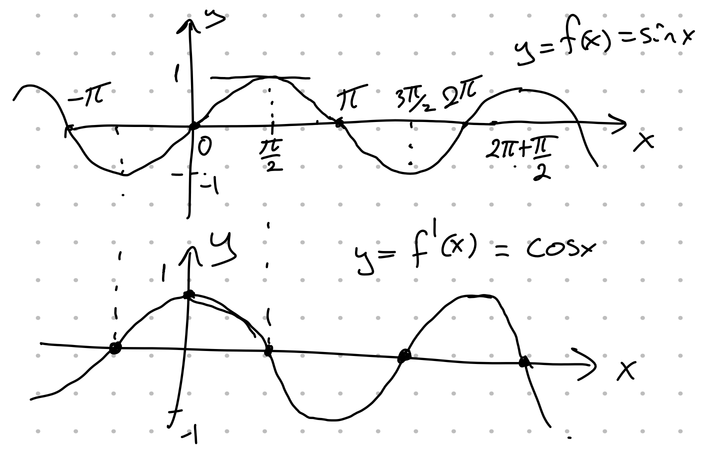
The quotient rule is not very efficient! It is better if we can avoid it.

Note $\quad F(x) = \boxed{\frac{3x^2}{x}} + \frac{2x^{1/2}}{x} = 3x + 2x^{-1/2}$

$$F'(x) = 3 + \cancel{2}\left(-\frac{1}{\cancel{2}}x^{-3/2}\right) = 3 - x^{-3/2}$$

3.3 Derivatives of Trigonometric Functions

$f(x) = \sin x$



$$\boxed{(\sin x)' = \cos x}$$

Example $y = x^2 \sin x$ find y'.

$$y' = 2x\cdot\sin x + x^2\cos x$$