Example Find f' if $f(x)=\frac{1-x}{2+x}$

$$f'(x)=\lim_{h\to0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to0}\frac{\frac{1-(x+h)}{2+(x+h)}-\frac{1-x}{2+x}}{h}$$

$$=\lim_{h\to0}\frac{1}{h}\cdot\left(\frac{1-x-h}{2+x+h}+\frac{x-1}{2+x}\right)=\lim_{h\to0}\frac{1}{h}\left(\frac{(1-x-h)(2+x)+(x-1)(2+x+h)}{(2+x+h)(2+x)}\right)$$

$$=\lim_{h\to0}\frac{1}{h}\left(\frac{\cancel{2}\cancel{-2x}-2h\cancel{+x}\cancel{-x^2}\cancel{-hx}\cancel{+2x}\cancel{+x^2}\cancel{+hx}\cancel{-2}\cancel{-x}-h}{(2+x+h)(2+x)}\right)$$

$$=\lim_{h\to0}\frac{-3\cancel{h}}{\cancel{h}(2+x+h)(2+x)}=\frac{-3}{(2+x)(2+x)}=\frac{-3}{(2+x)^2}$$

Other Notations

$$\boxed{f'(x)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=\lim_{\Delta x\to0}\frac{\Delta y}{\Delta x}}$$

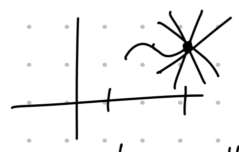
$$\underbrace{f'(x)=y'=\frac{dy}{dx}=\frac{df}{dx}=\frac{d}{dx}}_{\text{Leibniz's notation}}f(x)=Df(x)=D_xf(x)$$

$\frac{d}{dx}$, D : "differentiation operators"

$$f'(5)=\left.\frac{dy}{dx}\right|_{x=5}$$

Def^n f is differentiable at a if $f'(a)$ exists.

f is differentiable on (a,b) if $f'(x)$ exists for all $x\in(a,b)$



Example Where is the function $f(x)=|x|$ differentiable?

for $x>0$, $f(x)=|x|=x$ $\quad f'(x)=\lim_{h\to0}\frac{f(x+h)-f(x)}{h}$

$$=\lim_{h\to0}\frac{\cancel{x}+h\cancel{-x}}{h}=\lim_{h\to0}1=1$$

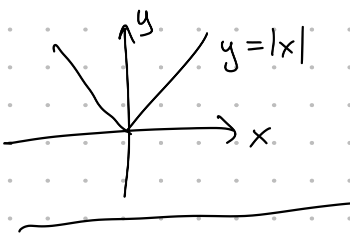
for $x<0$, $f(x)=|x|=-x$

$$f'(x)=\lim_{h\to0}\frac{-(x+h)-(-x)}{h}=\lim_{h\to0}\frac{\cancel{-x}-h\cancel{+x}}{h}=\lim_{h\to0}-1=-1$$

$$f'(0)=\lim_{h\to0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to0}\frac{|h|}{h}\quad \text{DNE}$$

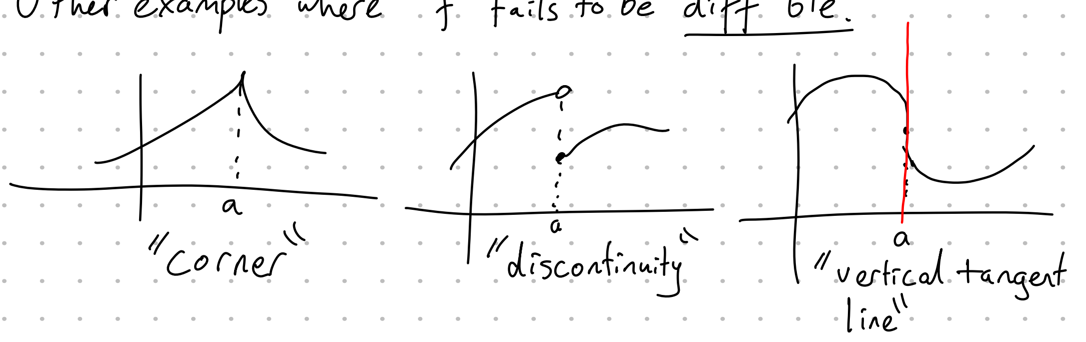
$$\lim_{h\to0^+}\frac{|h|}{h}=\lim_{h\to0^+}\frac{\cancel{h}}{\cancel{h}}=1\quad\text{but}\quad\lim_{h\to0^-}\frac{|h|}{h}=\lim_{h\to0^-}\frac{-\cancel{h}}{\cancel{h}}=-1$$

$f(x)=|x|$ at $x=0$ is not differentiable.



Thm If f is differentiable at a, then f is continuous a.

Other examples where f fails to be diff'ble.



High Order Derivatives

Since $f'(x)$ is "just another function", we can take its derivative to get $(f'(x))'=f''(x)$ called the second derivative of f, and ingeneral we can continue to take higher order derivatives to get $f'''(x)$, $f^{(4)}(x)$, $f^{(5)}(x)$ and so on.

In Leibniz notation: $f''(x)=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d^2y}{dx^2}$

Example If $f(x)=x^3-x$, find all higher order derivatives of f.

Recall from last time that $f'(x)=3x^2-1$

$$f''(x)=\lim_{h\to0}\frac{f'(x+h)-f'(x)}{h}=\lim_{h\to0}\frac{3(x+h)^2-1-(3x^2-1)}{h}$$

$$=\lim_{h\to0}\frac{\cancel{3x^2}+6xh+3h^2\cancel{-1}\cancel{-3x^2}\cancel{+1}}{h}=\lim_{h\to0}\frac{\cancel{h}(6x+3h)}{\cancel{h}}=6x$$

$f''(x)=6x$

$$f'''(x)=\lim_{h\to0}\frac{f''(x+h)-f''(x)}{h}=\lim_{h\to0}\frac{\cancel{6x}+6h\cancel{-6x}}{h}=6$$

$$f^{(4)}(x)=\lim_{h\to0}\frac{6-6}{h}=0\qquad f^{(5)}(x)=f^{(6)}(x)=\cdots=0$$

Chp 3 Differentiation Rules

3.1 Derivatives of Polynomials and Exponential Functions

If c is a constant $\frac{d(c)}{dx}=0$

$$\frac{d}{dx}(x)=1\qquad\frac{d(x^2)}{dx}=2x\qquad\frac{d(x^3)}{dx}=3x^2$$

The Power Rule If n is a positive integer, then

$$\frac{d}{dx}(x^n) = n x^{n-1}$$

Pf First note that

$$x^n - a^n = (x-a)(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \cdots + x^2 a^{n-3} + x a^{n-2} + a^{n-1})$$

$$= x^n + \cancel{x^{n-1}a} + \cancel{x^{n-2}a^2} + \cdots + \cancel{x^2 a^{n-2}} + \cancel{x a^{n-1}}$$

$$- (\cancel{x^{n-1}a} + \cancel{x^{n-2}a^2} + \cdots + \cancel{x^2 a^{n-2}} + \cancel{x a^{n-1}} + a^n)$$

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{x^n - a^n}{x - a}$$

$$= \lim_{x \to a} x^{n-1} + x^{n-2}a + \cdots + x a^{n-2} + a^{n-1}$$

$$= a^{n-1} + a^{n-2}a + \cdots + a \cdot a^{n-2} + a^{n-1} = n a^{n-1}$$

$$f'(x) = n x^{n-1} = (x^n)'$$

Thm Actually the same rule holds for any real number n.

So $(x^n)' = n x^{n-1}$.

e.g. $f(x) = x^6 \quad f'(x) = 6x^5 \qquad y = x^{1000} \quad y' = 1000 x^{999}$

$$\frac{d}{dx}\sqrt{x} = \frac{d}{dx}(x)^{1/2} = \frac{1}{2} x^{\frac{1}{2}-1} = \frac{1}{2} x^{-1/2} = \frac{1}{2x^{1/2}} = \frac{1}{2\sqrt{x}}$$

Example Differentiate:

a) $f(x) = \frac{1}{x^2} = x^{-2}$

$$f'(x) = -2x^{-2-1} = -2x^{-3}$$

b) $y = \sqrt[3]{x^2} = (x^2)^{1/3} = x^{2/3}$

$$y' = \frac{2}{3} x^{\frac{2}{3}-1} = \frac{2}{3} x^{-1/3}$$

c) $y = x\sqrt{x} = x\,x^{1/2} = x^{1+1/2} = x^{3/2}$

$$y' = \frac{3}{2} x^{1/2}$$

New derivatives from Old

$$\frac{d}{dx}(c f(x)) = c \frac{d}{dx}(f(x))$$

Pf: define $g(x) = c f(x)$

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$$= \lim_{h \to 0} \frac{c f(x+h) - c f(x)}{h} = c \underbrace{\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}} = c f'(x)$$

$$(c f(x))' = c f'(x)$$

Example

$$\frac{d}{dx}(3x^4) = 3 \frac{d}{dx}(x^4) = 3(4x^3) = 12x^3$$

$$\frac{d}{dx}(-x) = \frac{d}{dx}((-1)\cdot x) = (-1)\cdot\frac{d}{dx}(x) = (-1)\cdot 1 = -1$$

Thm

$$\frac{d}{dx}(f \pm g) = \frac{df}{dx} \pm \frac{dg}{dx}$$

Example $\frac{d}{dx}(x^8 + 12x^5 - 4x^4 + 10x^3 - 6x + 5)$

$$= \frac{d}{dx}x^8 + 12\frac{d}{dx}x^5 - 4\frac{d}{dx}(x^4) + 10\frac{d}{dx}x^3 - 6\frac{d}{dx}(x) + \cancel{\frac{d5}{dx}}^{\,0}$$

$$= 8x^7 + 12(5x^4) - 4(4x^3) + 10(3x^2) - 6$$