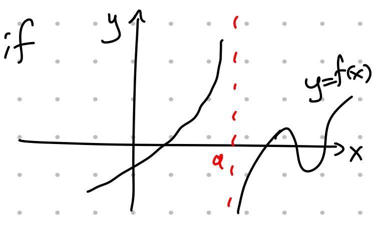
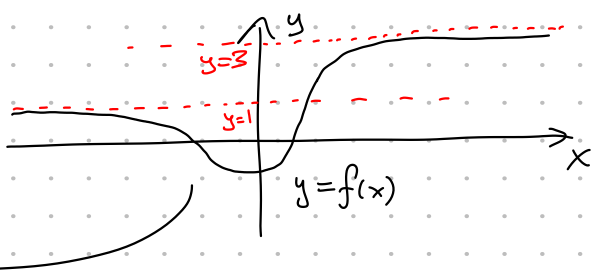
2.6 Limits at Infinity; Horizontal Asymptotes

Recall: Vert. asymp

if



Horizontal Asymptote:



$$\lim_{x\to\infty} f(x) = 3 \qquad \lim_{x\to -\infty} f(x) = 1$$

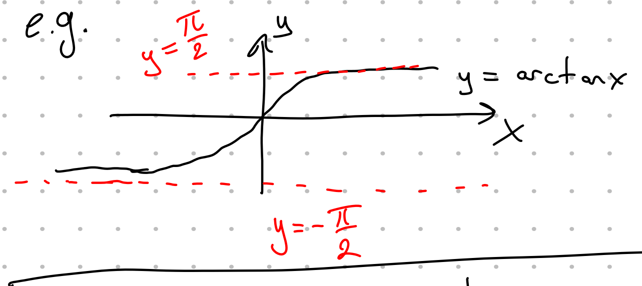
Intuitive Def^n of a limit at ∞

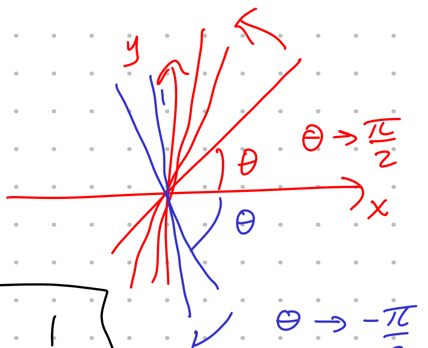
let f be a function defined on (a,∞). Then

$\lim\limits_{x\to\infty} f(x) = L$ means that the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large.

Def^n $y=L$ is called a horizontal asymptote of $y=f(x)$ if either $\lim\limits_{x\to\infty} f(x) = L$ or $\lim\limits_{x\to-\infty} f(x) = L$

e.g.





Example Find $\lim\limits_{x\to\infty} \frac{1}{x}$ and $\lim\limits_{x\to-\infty} \frac{1}{x}$

$$\frac{1 \leftarrow \text{const}}{x \leftarrow \text{very large number}} \approx 0 \qquad \lim_{x\to\infty} \frac{1}{x} = 0$$

similarly $\lim\limits_{x\to-\infty} \frac{1}{x} = 0$

Example Find $\lim\limits_{x\to\infty} \frac{3x^2-x-2}{5x^2+4x+1}$ the highest order term is x^2 so we divide both the numerator and the denominator by x^2.

$$= \lim_{x\to\infty} \frac{\frac{1}{x^2}(3x^2-x-2)}{\frac{1}{x^2}(5x^2+4x+1)} = \lim_{x\to\infty} \frac{\left(3\frac{\cancel{x^2}}{\cancel{x^2}} - \frac{x}{x^2} - \frac{2}{x^2}\right)}{\left(\frac{5\cancel{x^2}}{\cancel{x^2}} + \frac{4x}{x^2} + \frac{1}{x^2}\right)}$$

$$= \lim_{x\to\infty} \frac{3 - \overset{0}{\cancel{\frac{1}{x}}} - \overset{0}{\cancel{\frac{2}{x^2}}}}{5 + \underset{0}{\cancel{\frac{4}{x}}} + \underset{0}{\cancel{\frac{1}{x^2}}}} = \frac{3}{5}$$

Example Find the horizontal asymptotes of $f(x) = \frac{\sqrt{2x^2+1}}{3x-5}$

$$\lim_{x\to\infty} \frac{\sqrt{2x^2+1}}{3x-5} \qquad \sqrt{x^2} \sim x \text{ is the highest order term}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x}\left(\sqrt{2x^2+1}\right)}{3-\frac{5}{x}} = \lim_{x\to\infty} \frac{\sqrt{\frac{2x^2+1}{x^2}}}{3-\frac{5}{x}} = \lim_{x\to\infty} \frac{\sqrt{2+\overset{0}{\cancel{\frac{1}{x^2}}}}}{3-\underset{0}{\cancel{\frac{5}{x}}}}$$

$$= \frac{\sqrt{2}}{3}$$

$$\lim_{x\to-\infty} \frac{\sqrt{2x^2+1}}{3x-5} \qquad \left(\frac{1}{-x}\right)$$

$$\boxed{\begin{matrix} \text{If } A<0 & A=\frac{1}{x} \\ \sqrt{A^2}=-A & \sqrt{\frac{1}{x^2}} = -\frac{1}{x} \end{matrix}}$$

$$= \lim_{x\to-\infty} \frac{\left(-\frac{1}{x}\right)}{\left(-\frac{1}{x}\right)} \frac{\sqrt{2x^2+1}}{(3x-5)} = \lim_{x\to-\infty} \frac{\sqrt{\frac{2x^2+1}{x^2}}}{-3+\frac{5}{x}} = \lim_{x\to-\infty} \frac{\sqrt{2+\overset{0}{\cancel{\frac{1}{x^2}}}}}{-3+\underset{0}{\cancel{\frac{5}{x}}}}$$

$$= \frac{\sqrt{2}}{-3}$$

Example Compute $\lim\limits_{x\to\infty} \left(\sqrt{x^2+3x+1} - x\right)$ $\qquad (a-b)(a+b)=a^2-b^2$

$$= \lim_{x\to\infty} \frac{\left(\sqrt{x^2+3x+1}-x\right)\left(\sqrt{x^2+3x+1}+x\right)}{\left(\sqrt{x^2+3x+1}+x\right)} = \lim_{x\to\infty} \frac{(\cancel{x^2}+3x+1)-\cancel{x^2}}{\sqrt{x^2+3x+1}+x}$$

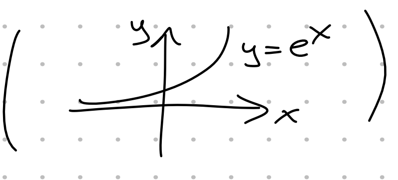
$$= \lim_{x\to\infty} \frac{3+\overset{0}{\cancel{\frac{1}{x}}}}{\sqrt{\underset{1}{\frac{\cancel{x^2}}{\cancel{x^2}}} + \underset{0}{\cancel{\frac{3x}{x^2}}} + \underset{0}{\cancel{\frac{1}{x^2}}}} + \frac{\cancel{x}}{\cancel{x}}1} = \frac{3}{\sqrt{1}+1} = \frac{3}{2}$$

Example Find $\lim\limits_{x\to 2^+} \arctan\left(\frac{1}{x-2}\right)$ $\qquad$ Set $t = \frac{1}{x-2}$

then $\lim\limits_{x\to 2^+} t = \lim\limits_{x\to 2^+} \frac{1}{x-2} = \infty$

$$\lim_{x\to 2^+} \arctan\left(\frac{1}{x-2}\right) = \lim_{t\to\infty} \arctan(t) = \frac{\pi}{2}$$

Recall $\boxed{\lim\limits_{x\to-\infty} e^x = 0}$

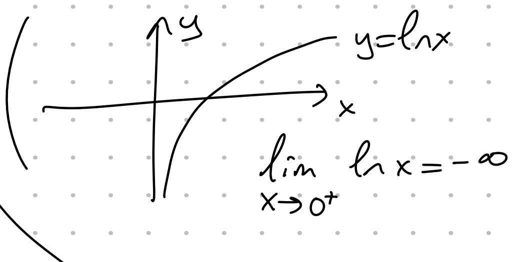


e.g. $\lim\limits_{x\to 6^+} \ln(x-6)$

$t = x-6 \qquad \lim\limits_{x\to 6^+} t = 0$

and $t>0$ since $x>6$

$$\lim_{x\to 6^+} \ln(x-6) = \lim_{t\to 0^+} \ln(t) = -\infty$$



$\lim\limits_{x\to 0^+} \ln x = -\infty$

Example Find $\lim_{x\to 0^-} e^{1/x} = 0$

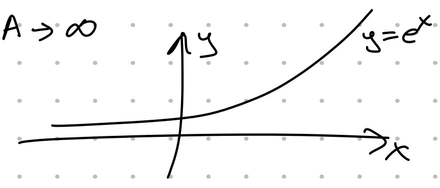
$\frac{1}{\boxed{x\to 0^-}}$ large negative $\frac{1}{x}\to -\infty$

since $\frac{1}{x}\to -\infty$.

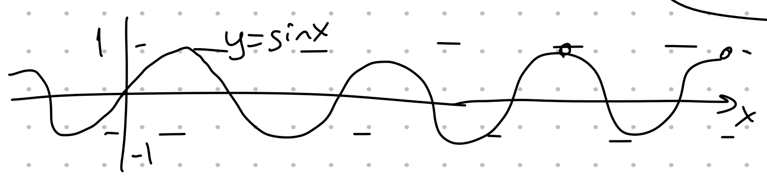
Find $\lim_{x\to 0^+} e^{1/x} = \infty$

$\frac{1}{x}\to +\infty$ as $x\to 0^+$

since $\frac{1}{x}\to\infty$ as $x\to\infty$ and $\lim_{A\to\infty} e^{A} = \infty$



Example Find $\lim_{x\to\infty} \sin x$



sinx keeps oscilating between -1 and 1.

$\lim_{x\to\infty} \sin x$ DNE.

Example Find $\lim_{x\to\infty} \frac{x^2+x}{3-x}$

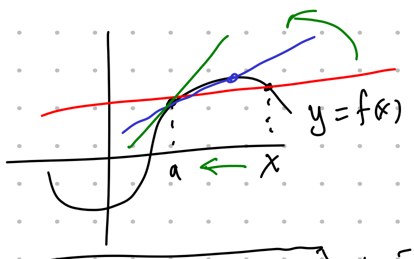
$\xrightarrow{\text{highestorder:}} x^2$

$\longrightarrow x$ ← we choose not x^2

$= \lim_{x\to\infty} \frac{(\frac{1}{x})(x^2+x)}{(\frac{1}{x})(3-x)} = \lim_{x\to\infty} \frac{x+1}{\boxed{\frac{3}{x}-1}}$ $\rightarrow +\infty$ $\rightarrow -1$ as $x\to\infty$

$= \frac{+\infty}{-1} = -\infty$

2.7 Derivatives and Rates of Change



Def^n The tangent line to the curve $y=f(x)$ at the point $P(a,f(a))$ is the line through P with slope $m = \lim_{x\to a} \underbrace{\frac{f(x)-f(a)}{x-a}}_{\text{slope of the secant line.}}$

Example Find an equation of the tangent line to $y=x^2$ at $P(1,1)$

$f(x)=x^2$

$m = \lim_{x\to 1} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} \frac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}}$

$=2$.

$y-y_0 = m(x-x_0) \rightarrow y-1 = 2(x-1)$

$y = 2x-2+1 = 2x-1$

$y=2x-1$