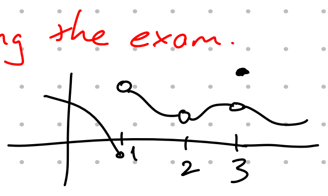
Check your email! You need to fill out a form indicating when you will be taking the exam.

Continuity: $\lim_{x \to a} f(x) = f(a)$



Example 2

Where are each of the following functions discontinuous?

a) $f(x) = \dfrac{x^2 - x - 2}{x - 2}$

$f(2)$ is not defined

f is not cont. at 2

b) $f(x) = \begin{cases} \frac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

$f(0) = 1$

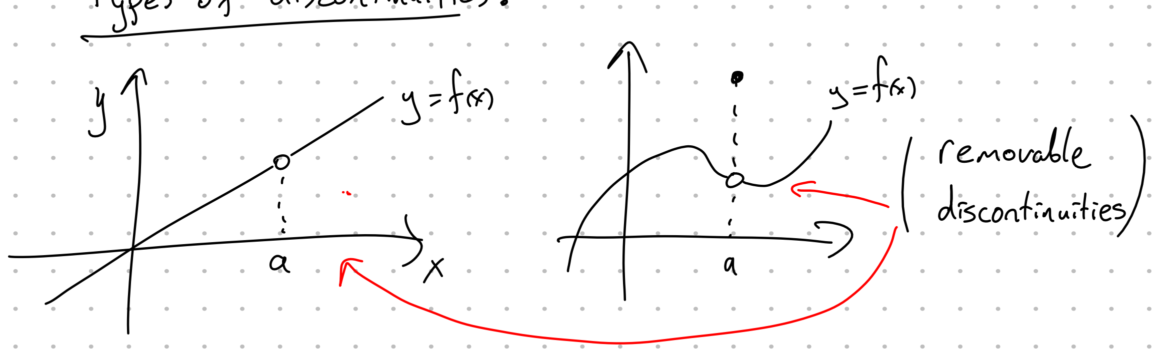
$\lim_{x \to 0} f(x) = \infty$ (DNE)

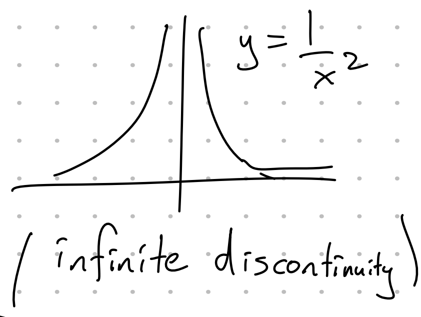
c) $f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$

$f(2) = 1$ $\quad \lim_{x \to 2} \dfrac{x^2 - x - 2}{x - 2} = \lim_{x \to 2} \dfrac{\cancel{(x-2)}(x+1)}{\cancel{(x-2)}} = 3 \neq 1 = f(2)$

f is not cont. at $x = 2$.

Types of discontinuities:





Def^n A function f is continuous on an interval if it is continuous at every number in the interval.

Example Show that the function $f(x) = 1 - \sqrt{1 - x^2}$ is cont. on $(-1, 1)$.

If $-1 < a < 1$ then (We want to show $\lim_{x \to a} f(x) = f(a)$)

$\lim_{x \to a} f(x) = \lim_{x \to a} 1 - \sqrt{1 - x^2} = 1 - \lim_{x \to a} \sqrt{1 - x^2}$

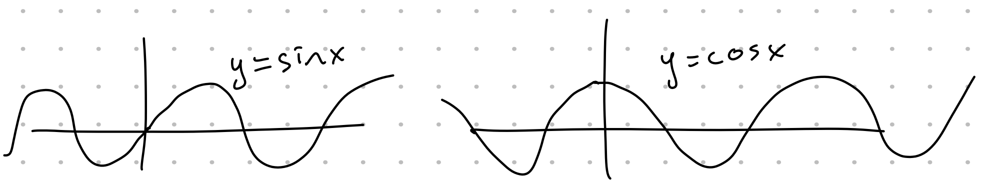
$= 1 - \sqrt{\boxed{\lim_{x \to a} 1 - x^2}} = 1 - \sqrt{1 - a^2} = f(a)$

Thm If f and g are cont. at a and c is a constant, then the following functions are cont. at a:

1) $f \pm g$ 2) cf 3) fg 4) $\dfrac{f}{g}$ if $g(a) \neq 0$.

Thm a) Any polynomial is cont. everywhere $\mathbb{R} = (-\infty, \infty)$

b) Any rational function is cont. everywhere it is defined, i.e. its domain.



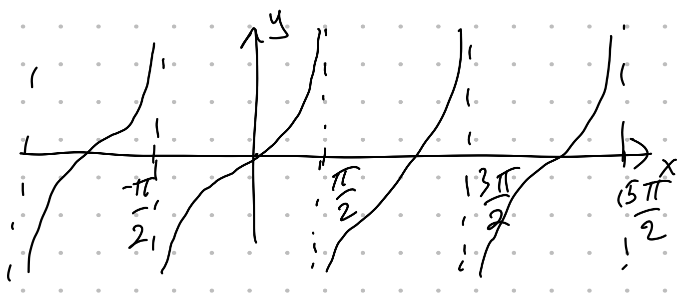
these are both cont. everywhere.

So $\tan x = \dfrac{\sin x}{\cos x}$ is cont. everywhere except for $\cos x = 0$

$\cos x = 0$ for $x = \ldots, -\frac{5\pi}{2}, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$

(odd multiples of $\frac{\pi}{2}$)

In other words $\tan x$ is cont everywhere in its domain.



Thm The following types of functions are cont. everywhere in their domain:

- polynomials
- rational functions
- root functions
- trig function
- inverse trigs
- exponentials
- logarithmic

Example Where is the function $f(x) = \dfrac{\ln x + \tan^{-1} x}{x^2 - 1}$ cont.?

$f(x)$ is cont. everywhere in its domain and the domain of $f(x)$ is $(0,1) \cup (1, \infty)$.

Note we have to exclude $x \leq 0$ since we have $\ln x$ and $x^2 = 1$ or $x = \pm 1$ since it is in the denom.

Example Find $\lim_{x\to\pi} \frac{\sin x}{2+\cos x}$

Since $-1 \le \cos x \le 1$ $\qquad \overset{1}{\overbrace{2-1}} \le 2+\cos x \le \overset{3}{\overbrace{2+1}}$

In particular $2+\cos x$ is never 0. So the function $\frac{\sin x}{2+\cos x}$ is cont. everywhere. $\lim_{x\to\pi} \frac{\sin x}{2+\cos x} = \frac{\sin \pi}{2+\cos\pi} = \frac{0}{2-1} = 0$

Thm If f is cont. at b and $\lim_{x\to a} g(x) = b$ then $\lim_{x\to a} f(g(x)) = f(b)$. In other words,

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right)$$

Example Find $\lim_{x\to 1} \arcsin\left(\frac{1-\sqrt{x}}{1-x}\right)$

$$= \arcsin\left(\boxed{\lim_{x\to 1} \frac{1-\sqrt{x}}{1-x}}\right) = \arcsin\left(\tfrac{1}{2}\right) = \frac{\pi}{6}$$

$$\lim_{x\to 1} \frac{1-\sqrt{x}}{1-x} = \lim_{x\to 1} \frac{(1-\sqrt{x})}{(1-x)}\frac{(1+\sqrt{x})}{(1+\sqrt{x})} = \lim_{x\to 1} \frac{\cancel{1-x}}{\cancel{(1-x)}(1+\sqrt{x})}$$

$$= \lim_{x\to 1} \frac{1}{1+\sqrt{x}} = \frac{1}{2}$$

Thm If g is cont. at a and f is cont. at $g(a)$, then the composite function $f\circ g$ given by $(f\circ g)(x) = f(g(x))$ is cont. at a.

Example Where are the following functions cont?

a) $h(x) = \sin(x^2)$

$u = x^2$, $v = \sin x$ — both cont. everywhere

So $\sin(x^2)$ is cont. everywhere

b) $F(x) = \ln(1+\cos x)$

$1+\cos x$ cont. everywhere

$\ln x$ cont. on $(0,\infty)$

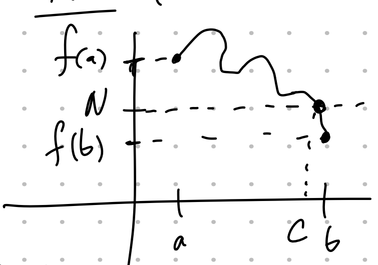
So $\ln(1+\cos x)$ is cont as long as $1+\cos x > 0$

$1+\cos x = 0$ when $\cos x = -1$

when $x = \ldots -3\pi, -\pi, \pi, 3\pi \ldots$ (odd multiples of π)

$F(x)$ is cont. for $x \ne \ldots, -3\pi, -\pi, \pi, 3\pi, \ldots$

Thm (The Intermediate Value Theorem) (IVT)



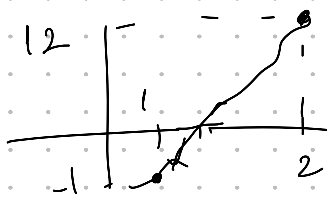
there is c s.t. $f(c) = N$

Suppose that f is cont. on $[a,b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \ne f(b)$. Then there exists a number c in (a,b) s.t. $f(c) = N$

Example Show that there is a root of the function $f(x) = 4x^3 - 6x^2 + 3x - 2 = 0$ between 1 and 2

$f(1) = 4 - 6 + 3 - 2 = -1 < 0$

$f(2) = 4(2)^3 - 6(2)^2 + 3(2) - 2 = 12 > 0$



Since $f(1) < 0 < f(2)$ and f is cont., there is c in $(1,2)$ s.t. $f(c) = 0$