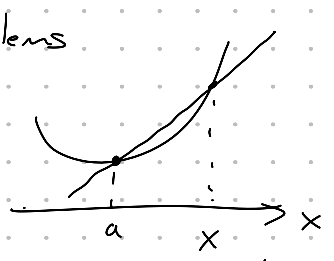
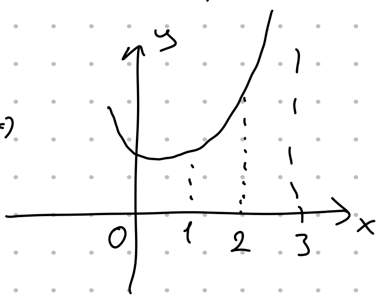
Last time: Tangent and Velocity Problems

They are of the form $\lim_{x \to a} \frac{f(x)-f(a)}{x-a}$



Today: 2.2 The limit of a function

$\lim_{x \to a} g(x)$ e.g. $\lim_{x \to 2} (x^2-x+2) = f(x)$



x	$f(x)$
1.0	2
1.5	2.75
1.9	3.71
1.999	3.997001

x	$f(x)$
3.0	8
2.5	5.75
2.1	4.31
2.001	4.003001

$\lim_{x \to 2} (x^2-x+2) = 4$ (it happens for this example that $f(2)=4$)

Def^n of a limit

Suppose $f(x)$ is defined when x is in some open interval that contains a except possibly at a itself. Then we write

$\lim_{x \to a} f(x) = L$ if we can make the values of $f(x)$ arbitrarily close to L by restricting x to be sufficiently close to a.

Notation: $\lim_{x \to a} f(x) = L \iff f(x) \to L$ as $x \to a$

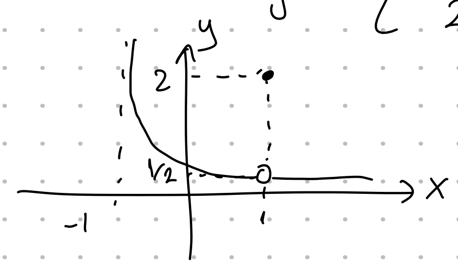
Example Guess the value of $\lim_{x \to 1} \frac{x-1}{x^2-1}$

$x<1$	$f(x)$
0.9	0.526316
0.999	0.500250

$x>1$	$f(x)$
1.1	0.476190
1.001	0.499750

$\lim_{x \to 1} \frac{x-1}{x^2-1} = 0.5$

Say we define $g(x) = \begin{cases} \frac{x-1}{x^2-1} & \text{if } x \neq 1 \\ 2 & \text{if } x=1 \end{cases}$



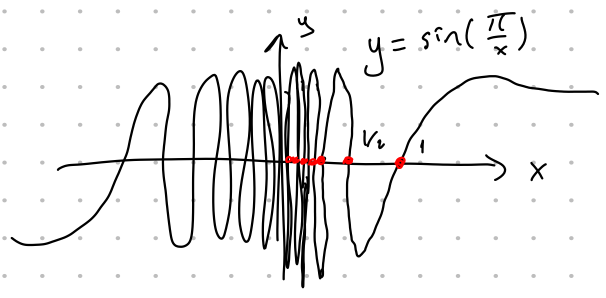
since $g(x)$ and $\frac{x-1}{x^2-1}$ agrees for $x \neq 1$, their limits at $x=1$ are the same.

$\lim_{x \to 1} g(x) = \lim_{x \to 1} \frac{x-1}{x^2-1} = 0.5$

Example Investigate $\lim_{x \to 0} \sin \frac{\pi}{x}$

$f(1) = \sin(\pi) = 0$ $\quad f(\frac{1}{3}) = \sin(\frac{\pi}{\frac{1}{3}}) = \sin(3\pi) = 0$

$f(0.1) = \sin(10\pi) = 0$ $\quad f(0.001) = \sin(1000\pi) = 0$

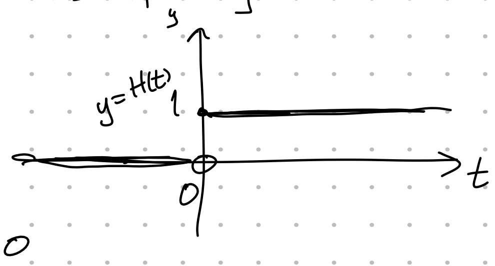


$\lim_{x \to 0} \sin(\frac{\pi}{x})$ does not exist (D.N.E)

One-Sided limits

Example The Heaviside function is defined by

$H(t) = \begin{cases} 0 & \text{if } t<0 \\ 1 & \text{if } t \geq 0 \end{cases}$



If we compute $H(t)$ for $t>0$ we always get 1. So we write $\lim_{t \to 0^+} H(t) = 1$

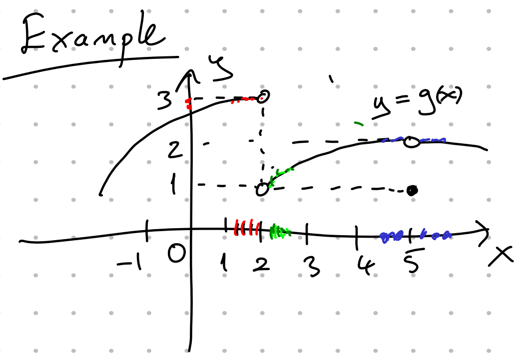
Similarly we have $\lim_{t \to 0^-} H(t) = 0$

Def^n $\lim_{x \to a^-} f(x) = L$ means that we make $f(x)$ arbitrarily close to L by take x sufficiently close to a with $x<a$.

Def^n $\lim_{x \to a^+} f(x) = L$ means that we make $f(x)$ arbitrarily close to L by take x sufficiently close to a with $x>a$.

$\lim_{x \to a} f(x) = L \iff \lim_{x \to a^+} f(x) = L$ and $\lim_{x \to a^-} f(x) = L$

Example



Find the following values

a) $\lim_{x \to 2^-} g(x) = 3$

b) $\lim_{x \to 2^+} g(x) = 1$

c) $\lim_{x \to 2} g(x)$ DNE

d) $g(2)$ DNE

e) $\lim_{x \to 5^-} g(x) = 2$

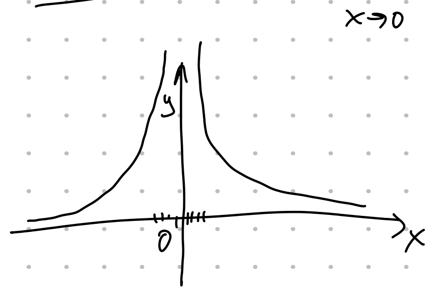
f) $\lim_{x \to 5^+} g(x) = 2$

g) $\lim_{x \to 5} g(x) = 2$

h) $g(5) = 1$

Infinite limits

Example Find $\lim\limits_{x\to 0} \frac{1}{x^2}$ if it exists.



∞ is not a number!

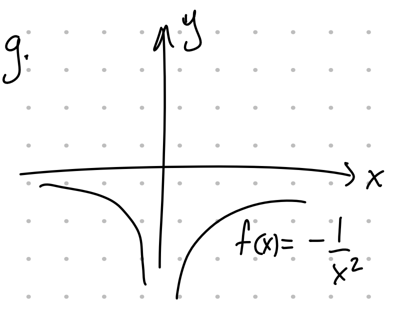
$$\boxed{\lim_{x\to 0} \frac{1}{x^2} = \infty}$$

is just a notation that says

the values of $\frac{1}{x^2}$ can be made arbitrarily large by taking x sufficiently close to 0.

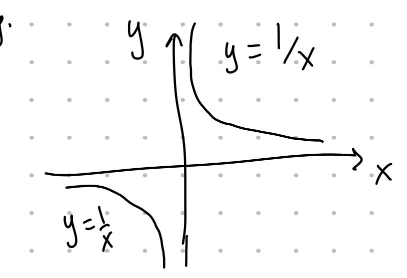
Actually we would say that $\lim\limits_{x\to 0} \frac{1}{x^2}$ DNE.

e.g.



$$\lim_{x\to 0} -\frac{1}{x^2} = -\infty$$

e.g.

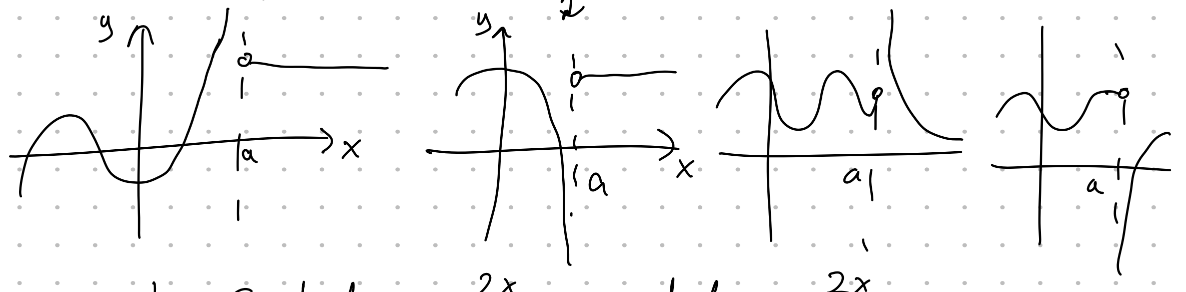


$$\lim_{x\to 0^+} \frac{1}{x} = \infty \qquad \lim_{x\to 0^-} \frac{1}{x} = -\infty$$

$$\lim_{x\to 0} \frac{1}{x} \text{ DNE}$$

Def^n The vertical line $x=a$ is called a vertical asymptote of the curve $y=f(x)$ if at least one of the following is true:

$\lim\limits_{x\to a^-} f(x) = \infty$, $\lim\limits_{x\to a^-} f(x) = -\infty$, $\lim\limits_{x\to a^+} f(x) = \infty$, or $\lim\limits_{x\to a^+} f(x) = -\infty$



Example Find $\lim\limits_{x\to 3^+} \frac{2x}{x-3}$ and $\lim\limits_{x\to 3^-} \frac{2x}{x-3}$

say $x\to 3^+$

$2x \to 6^+ > 0$

$x-3 \to 0^+ > 0$

$\Updownarrow$

$x > 3 \iff x-3 > 0$

$\frac{\text{positive const}}{\text{positive small \#}} = $ positive large number

So $\lim\limits_{x\to 3^+} \frac{2x}{x-3} = +\infty$

Say $x\to 3^-$

$2x \to 6^- > 0$

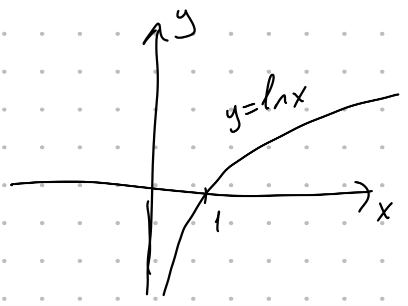
$x-3 \to 0^- < 0$

$\Updownarrow$

$x < 3 \iff x-3 < 0$

$\frac{\text{positive const}}{\text{neg. small \#}} = $ neg. large number

So $\lim\limits_{x\to 3^-} \frac{2x}{x-3} = -\infty$



so $x=0$ is a vertical asymptote of $\ln x$.

$$\lim_{x\to 0^+} \ln x = -\infty$$

2.3 Calculating Limits Using the Limit Laws

Limit Laws Suppose that c is a constant and the limits $\lim\limits_{x\to a} f(x)$ and $\lim\limits_{x\to a} g(x)$ exist. Then

$$\lim_{x\to a} (f(x) \pm g(x)) = \left(\lim_{x\to a} f(x)\right) \pm \left(\lim_{x\to a} g(x)\right)$$

$$\lim_{x\to a} (c f(x)) = c \lim_{x\to a} f(x)$$

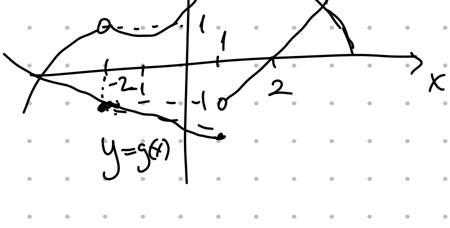
$$\lim_{x\to a} (f(x) g(x)) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$$

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \quad \text{if } \lim_{x\to a} g(x) \neq 0$$

$$\left(\lim_{x\to 0} \frac{x}{x} \neq \cancel{\frac{\lim_{x\to 0} x = 0}{\lim_{x\to 0} x = 0}} \right)$$

$$\lim_{x\to 0} 1 = 1$$

Example



Note $\lim\limits_{x\to -2} f(x) = 1 \quad \lim\limits_{x\to -2} g(x) = -1$

Find

a) $\lim\limits_{x\to -2} (f(x) + 5g(x)) = 1 + 5(-1) = -4$

b) $\lim\limits_{x\to 1} f(x) g(x)$

c) $\lim\limits_{x\to 2} \frac{f(x)}{g(x)}$