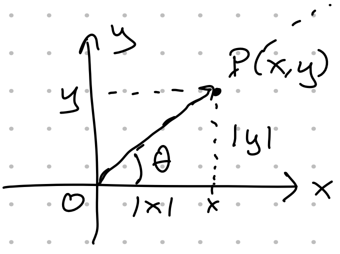
Trigonometric functions

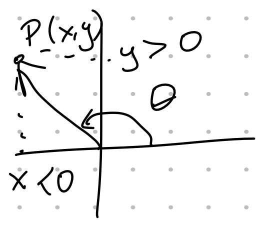


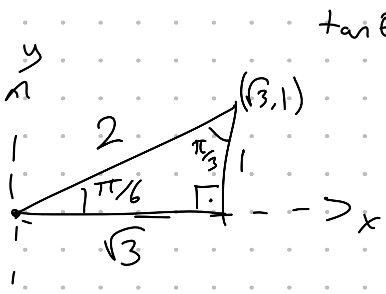
$r = \sqrt{x^2 + y^2}$

$\sin\theta = \frac{y}{r}$ $\quad \csc\theta = \frac{1}{\sin\theta}$

$\cos\theta = \frac{x}{r}$ $\quad \sec\theta = \frac{1}{\cos\theta}$

$\tan\theta = \frac{y}{x}$ $\quad \cot\theta = \frac{1}{\tan\theta}$

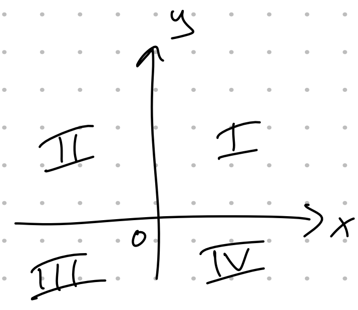




$\sin\frac{\pi}{6} = \frac{1}{2}$

$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

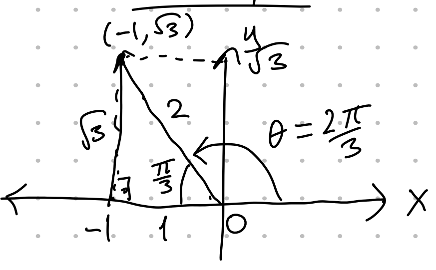
$\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$



$\sin\theta = \frac{y}{r} > 0$ in I and II

$\cos\theta = \frac{x}{r} > 0$ in I and IV

Example Find $\sin\theta$, $\cos\theta$, $\tan\theta$ for $\theta = \frac{2\pi}{3}$



$x = -1$
$y = \sqrt{3}$
$r = 2$

$\sin\left(\frac{2\pi}{3}\right) = \frac{y}{r} = \frac{\sqrt{3}}{2}$

$\cos\left(\frac{2\pi}{3}\right) = \frac{x}{r} = \frac{-1}{2}$

$\tan\left(\frac{2\pi}{3}\right) = \frac{y}{x} = \frac{\sqrt{3}}{-1} = -\sqrt{3}$

Trigonometric identities

① $\sin^2\theta + \cos^2\theta = 1$

$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} \rightarrow \left(\frac{\sin\theta}{\cos\theta}\right)^2 + 1 = \sec^2\theta$

Since $\tan\theta = \frac{\sin\theta}{\cos\theta}$,

② $\tan^2\theta + 1 = \sec^2\theta$

Similarly ③ $1 + \cot^2\theta = \csc^2\theta$

④ $\sin(-\theta) = -\sin\theta$ $\quad$ ⑤ $\cos(-\theta) = \cos\theta$

⑥ $\sin(\theta + 2\pi) = \sin\theta$ $\quad$ ⑦ $\cos(\theta + 2\pi) = \cos\theta$

Double-angle formulas:

⑧ $\sin 2x = 2\sin x\cos x$ $\quad$ these are obtained by using ①

⑨ $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$

Note that if $\cos 2x = 2\cos^2 x - 1$ then

$\cos 2x + 1 = 2\cos^2 x$ and

⑩ $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

similarly

⑪ $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

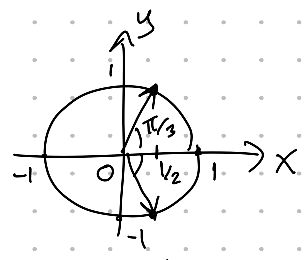
Example Find all values of x in the interval $[0, 2\pi]$ such that $\sin x = \sin 2x$

By ⑧, $\sin x = 2\sin x\cos x$ then

$0 = 2\sin x\cos x - \sin x = \sin x(2\cos x - 1)$

So either $\sin x = 0$ that is $x = 0, \pi, 2\pi$

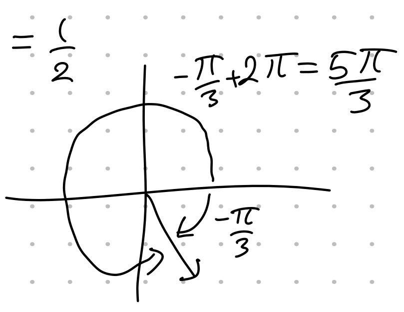
or $2\cos x - 1 = 0$ $\quad \boxed{\cos x = \frac{1}{2}}$ $\quad x = \frac{\pi}{3}, \frac{5\pi}{3}$

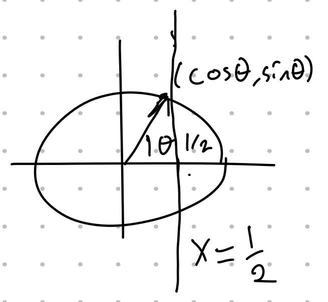


$\cos\theta = \frac{x}{r} = x = \frac{1}{2}$

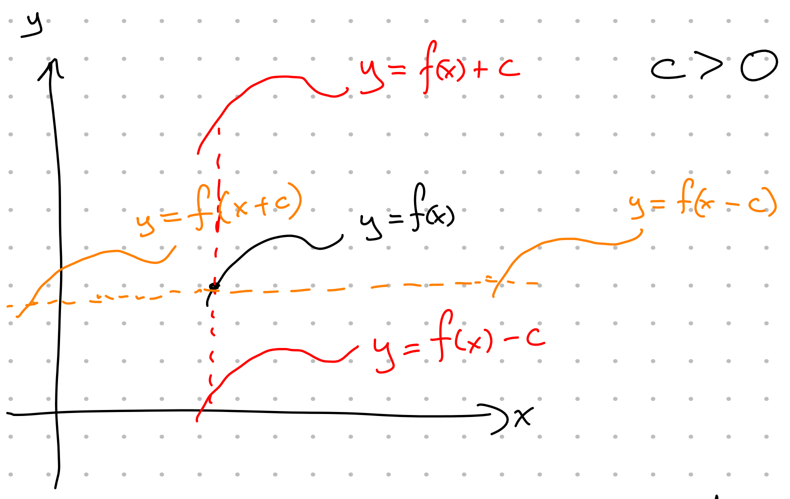
$r = 1$

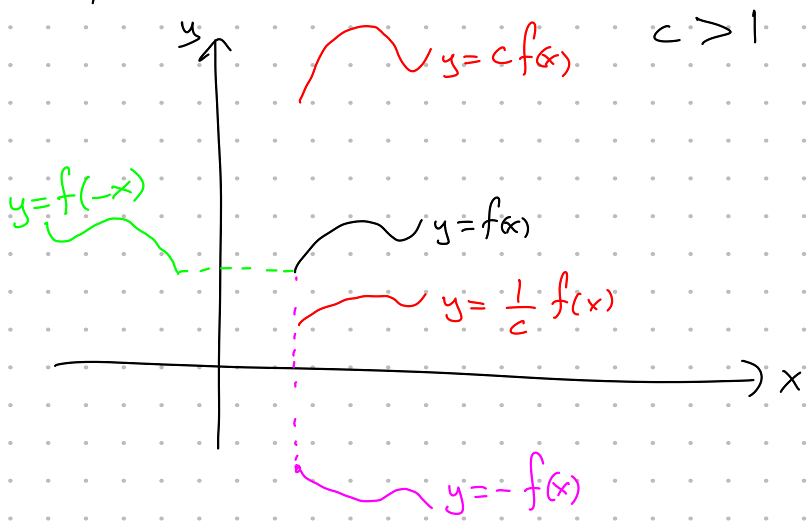
$\cos\theta = x$
$\sin\theta = y$





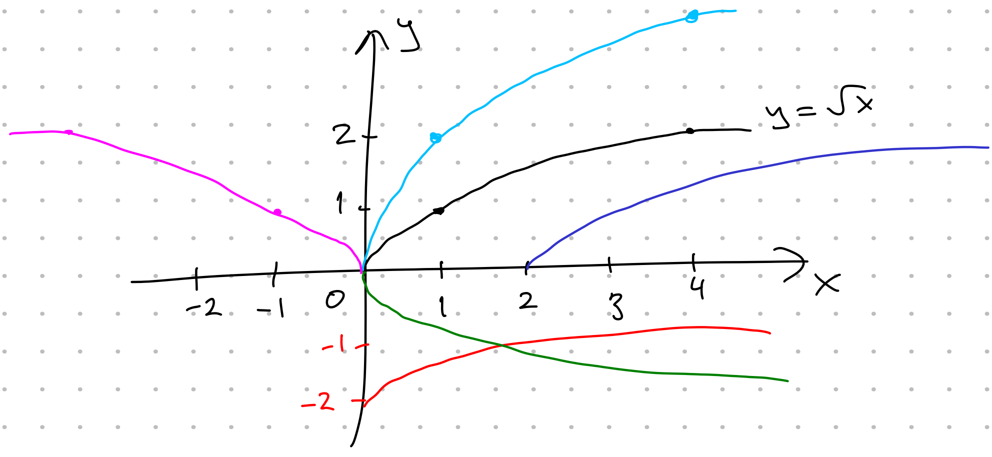
1.3 New functions from old functions





Example Given the graph of $y=\sqrt{x}$, sketch

$y=\sqrt{x}-2$ $\quad y=\sqrt{x-2}$, $y=-\sqrt{x}$, $y=2\sqrt{x}$, $y=\sqrt{-x}$



Combinations of functions

$$(f \pm g)(x) = f(x) \pm g(x) \qquad (fg)(x) = f(x)\,g(x)$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

The domain of the new functions ($f \pm g$, fg, $\frac{f}{g}$) is the intersection of the domains of f and g.
For $\frac{f}{g}$ we also require that $g(x) \neq 0$

Composition of functions:

Say $y = f(u) = \sqrt{u}$ and $u = g(x) = x^2+1$

So actually $y = \sqrt{u} = \sqrt{x^2+1}$ is a function of x as well. This is an example of composition of functions.

$$y = f(u) = f(g(x)) \qquad \text{we write } y = (f \circ g)(x) = f(g(x))$$

Example If $f(x) = x^2$ and $g(x) = x-3$ find $f \circ g$ and $g \circ f$.

$$(f \circ g)(x) = f(g(x)) = (g(x))^2 = (x-3)^2$$

$$(g \circ f)(x) = g(f(x)) = g(x^2) = x^2-3 \neq (x-3)^2$$