Wednesday, September 18, 2019 MATH 255 Notes

Notes about the homework:

Regarding Problem 4:

- In general, we say u and v are orthogonal if $u \cdot v = 0$ (in particular 0 is orthogonal to all vectors).
- However a normal vector to a plane has to be nonzero.

$$n \cdot (p - p_0) = 0$$

- Because of these facts, it is not enough to just plug in $(p-p_1) \times (p-p_2)$ for n_0 and $(p-p_3)$ for $(p-p_0)$ in the equation of the plane to show that $(p-p_1) \times (p-p_2) \cdot (p-p_3) = 0$ describes a plane.
- $Q(p) = 0$ is the equation of a plane if and only if all p in the plane satisfies $Q(p) = 0$ and all p not in the plane satisfies $Q(p) \neq 0$.
- For example, $Q(p) = p \times p = 0$ is satisfied in any plane, but also in all of $\mathbb{R}^3$, so it does not define a plane.

Regarding Weak tangents.

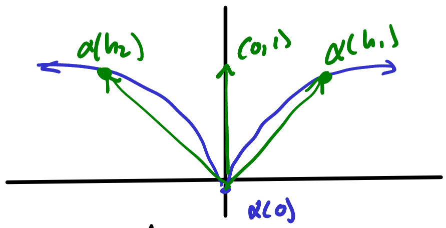
Consider $\alpha(t) = (t, t^2, t^3)$. What is the weak tangent at $t = 0$? Let L_h = line passing through $\alpha(h)$ and $\alpha(0)$. We may take $D_h = \dfrac{\alpha(h) - \alpha(0)}{|\alpha(h) - \alpha(0)|}$ to be the unit direction vector of L_h. To compute the weak tangent, we have

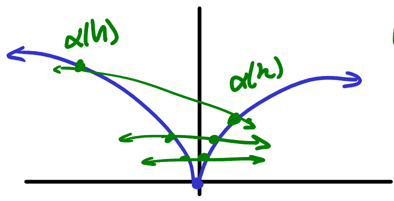
$$\lim_{h \to 0} D_h = \lim_{h \to 0} \frac{(h, h^2, h^3)}{\sqrt{h^2 + h^4 + h^6}} = \lim_{h \to 0} \frac{h}{\sqrt{h^2}} \frac{(1, h, h^2)}{\sqrt{1 + h^2 + h^4}} = \frac{h}{|h|} \frac{(1, h, h^2)}{\sqrt{1 + h^2 + h^4}}$$

Note that $\dfrac{h}{|h|} = \begin{cases} 1 & \text{if } h > 0 \\ -1 & \text{if } h < 0 \end{cases}$ Also, $D_h \to (1,0,0)$ when $h \to 0^+$ but $D_h \to (-1,0,0)$ when $h \to 0^-$

$\Rightarrow \lim_{h \to 0} D_h$ does not exist. Nevertheless, $(1,0,0)$ and $(-1,0,0)$ are equivalent as direction vectors since we allow multiplication by scalars.

In part a) of question 3, we had a curve like this:





where $k=-h$

In this case when we are finding a weak tangent, the lines we are considering always pass through $\alpha(0)$. Here, the limit <u>has</u> to be a vertical line.

When we compute the strong tangent on the other hand, from one direction the limit we get is a horizontal line (letting $k=-h$). However, the weak tangent case is a subcase of this one (when $h=0$). $\Rightarrow$ The limit of D_n does not exist <u>even when we allow scaling</u>.

Last time we wrote down

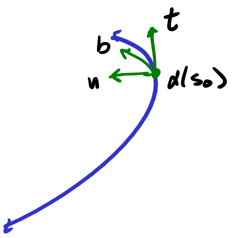
The Fundamental Theorem of the Local Theory of Curves:

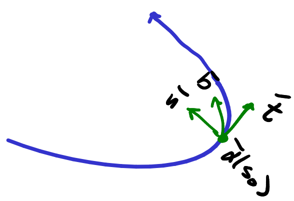
Given $k(s)>0$ and $\tau(s)>0$, $\exists$ $\alpha(s)$ whose curvature is $k(s)$ and torsion is $\tau(s)$. This α is unique up to rigid motions in $\mathbb{R}^3$.

We also went over the

Frenet Formulas:

$$\left.\begin{aligned} t' &= kn \\ n' &= kt - \tau b \\ b' &= \tau n \end{aligned}\right\} \text{ This is a } 9\times 9 \text{ linear system of differential equations.}$$





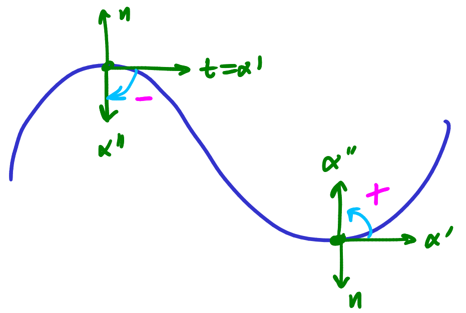
$t=\alpha'$

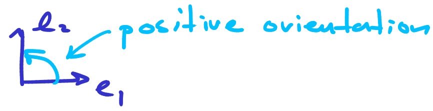
$\alpha=\int t\,ds$

Apply existence & uniqueness of ODEs.

Now we will talk about some global properties of curves.

Say $\alpha: I \to \mathbb{R}^2$. Then we can define a signed curvature $k: I \to \mathbb{R}$

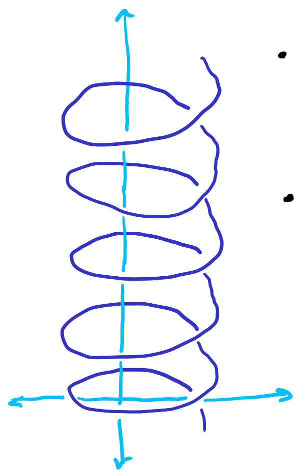




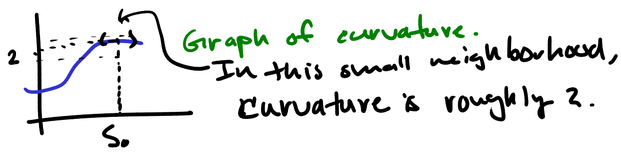
(assign $+$ to points where $\{\alpha'', \alpha'\}$ form a positive basis and $-$ to points where $\{\alpha'', \alpha'\}$ form a negative basis)

To make this idea rigorous, we define $n(s)$ to be a vector that makes $\{t(s)=\alpha'(s), n(s)\}$ a positively orthonormal basis. Then we define k by

$$\frac{dt}{ds} = kn.$$



- One of the exercises: Compute $k(s)$ and $\tau(s)$ for the helix. You will notice that they are both constant.
- What is the importance of that? Well, for an arbitrary curve, the curvature + torsion also are roughly constant locally



Reparametrization By Arc Length

Given a regular, parametrized curve $\alpha: I \to \mathbb{R}^3$, we can reparametrize it by arc length as follows:

First, define $\quad s = f(t) = \int_{t_0}^{t} |\alpha'(u)|\, du$

What can we say about $f'(t)$?

By the Fundamental Theorem of Calculus, $s' = f'(t) = |\alpha'(t)| > 0$ by assumption

$\Rightarrow$ f is an increasing function of t $\Rightarrow$ f is an invertible function & $t = f^{-1}(s)$.

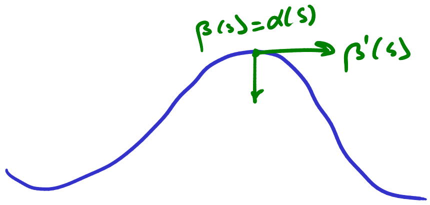
Therefore, we may define $\beta(s) = \alpha(f^{-1}(s))$.

Recall that $(f^{-1})'(s) = \dfrac{1}{f'(f^{-1}(s))}$.

By the chain rule,
$$\begin{aligned}\beta'(s) &= \alpha'(f^{-1}(s)) \cdot (f^{-1})'(s) \\ &= \alpha'(f^{-1}(s)) \cdot \frac{1}{f'(f^{-1}(s))} \\ &= \frac{\alpha'(t)}{f'(t)} = \frac{\alpha'(t)}{|\alpha'(t)|}\end{aligned}$$
$\Rightarrow \beta$ is a unit vector!

$\Rightarrow \beta$ is parametrized by arc length.

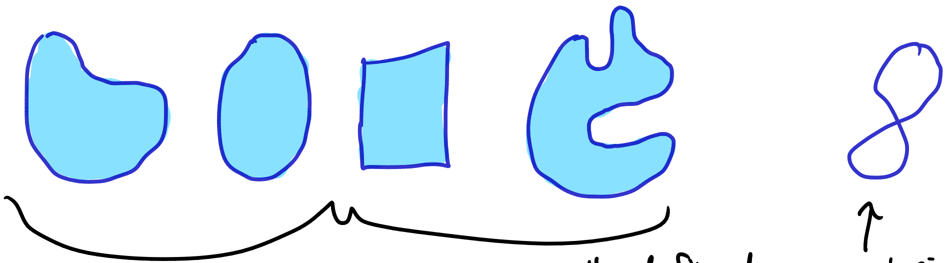
Exercise: Show that the trace of β is the same as the trace of α.



$|\alpha'(t)| > 0$

Next, the Isoperimetric Inequality (From section 1.7)

Question:



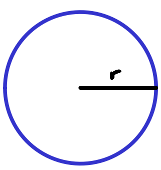
Simple, closed curves have well-defined interiors.

Not simple!

Among all simple closed curves of length (perimeter) ℓ, which one has the largest area? Answer: the circle!

$$\boxed{\text{Isoperimetric Inequality: } \ell - 4\pi A \geq 0}$$

For a circle,



$\ell = 2\pi r$

$A = \pi r^2$

We see that this equation is an equality for a circle.

The theorem we will state will tell us that although this inequality is always true for simple, closed curves, it becomes an equality only for circles!

Definition: A function f defined on $[a,b]$ is called differentiable if it can be extended to an open interval that contains $[a,b]$.