Wednesday, November 6, 2019 MTH255 Notes

Last Time: Let k_1, k_2 be the eigenvalues of dN, i.e. $\exists\, v_1, v_2$ such that $dN(v_1) = k_1 v_1$ and $dN(v_2) = k_2 v_2$. (We can choose v_1 and v_2 to be orthonormal and assume $k_1 \geq k_2$). Then k_1 and k_2 are called **principal curvatures** and v_1 + v_2 are called **principal directions**. Also

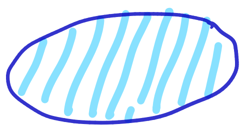
$$\left.\begin{array}{l} \textbf{Gaussian Curvature} = K = \det(-dN) = k_1 k_2 \\ \textbf{Mean Curvature} = H = -\frac{1}{2}\operatorname{tr}(dN) = \frac{1}{2}(k_1 + k_2) \end{array}\right\} \text{Pointwise}$$

Definition: $p \in S$ is called

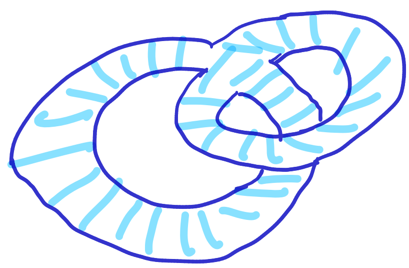
1) **Elliptic** if $\det(dN_p) = \det(-dN_p) = K > 0$
2) **Hyperbolic** if $\det(dN_p) < 0$
3) **Parabolic** if $\det(dN_p) = 0$ but $dN_p \neq 0$
4) **Planar** if $\det(dN_p) = 0$ but $dN_p = 0$

Definition: A surface S is called **minimal** if, for every $p \in S$ the mean curvature is zero.

Question: Among surfaces which have a given "wire frame" (Curve) as their boundary, which one has minimal area?



For example, if you have a circle, the minimal surface is a disk.

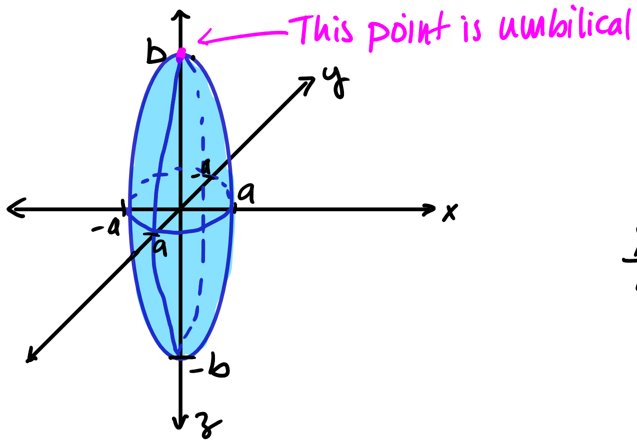


Of course, the problem is more interesting if your curve is more complicated.

- Type "minimal surface bubbles" into YouTube to see some examples. More on this later.

Definition: If, at $p \in S$, $k_1 = k_2$ then p is called an umbilical point of S. E.g. S^2 or a plane.

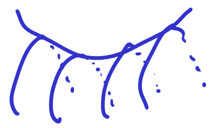
Proposition: If all points of a connected surface S are umbilical points then S is either contained in a sphere or a plane.

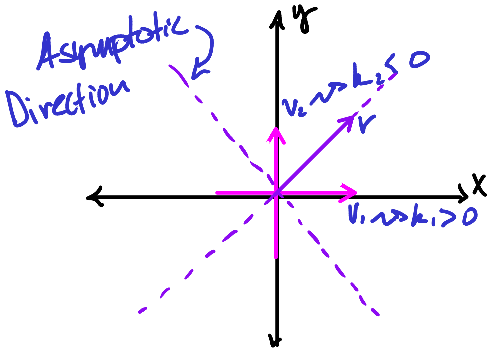


$$\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{b^2} = 1$$

Definition: Let $p \in S$. An asymptotic direction of S at p is a direction of T_pS for which the normal curvature $=0$. An asymptotic curve of S is a regular connected curve $C \subset S$ such that $\forall p \in C$, the tangent to C at p is an asymptotic direction: $k_2 \leq II_p(v) \leq k_1$

The points which are "not allowed" are hyperbolic:





If $v = av_1 + bv_2$ then

$II_p(v) = a^2k_1 + b^2k_2 = a^2 - b^2 = 0$ if $k_1 = 1, k_2 = -1$

Section 3.3: The Gauss map in Coordinates.

We know how to express the first fundamental form in local coordinates:

$I_p(aX_u + bX_v) = a^2E + 2abF + b^2G$ where $E = |X_u|^2$, $F = \langle X_u, X_v \rangle$, $G = |X_v|^2$

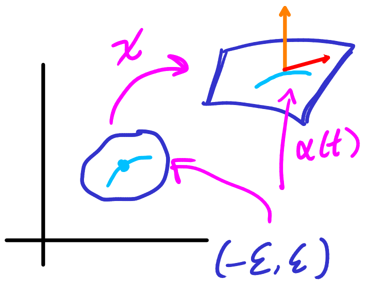
$II_p(aX_u + bX_v) = a^2e + 2abf + b^2g$ by virtue of being a quadratic form.

Question: What are e, f, and g ?

Recall: $\alpha' = u'X_u + v'X_v$ (because $\alpha(t) = X(u(t), v(t))$)

We use parametrizations $X : U \subset \mathbb{R}^2 \longrightarrow S$ such that $N = \dfrac{X_u \times X_v}{|X_u \times X_v|}$

$$dN(\alpha') = \frac{d}{dt} N(u(t), v(t)) = N_u u' + N_v v' \quad *$$



We know $N_u, N_v \in T_p S$, so we have

$$N_u = a_{11} X_u + a_{21} X_v$$

$$N_v = a_{12} X_u + a_{22} X_v$$

Substituting in $*$ we get

$$dN(u'X_u + v'X_v) = dN(\alpha') = (a_{11}X_u + a_{21}X_v)u' + (a_{12}X_u + a_{22}X_v)v'$$
$$= (a_{11}u' + a_{12}v')X_u + (a_{21}u' + a_{22}v')X_v$$

We now have a matrix representation of dN:

$$[dN]\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} a_{11}u' + a_{12}v' \\ a_{21}u' + a_{22}v' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} u' \\ v' \end{pmatrix}$$

$$\Rightarrow [dN] = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Careful: This matrix is not necessarily symmetric unless X_u, X_v are orthonormal.

- $II_p(\alpha') = -\langle dN(\alpha'), \alpha' \rangle = -\langle u'N_u + v'N_v, u'X_u + v'X_v \rangle$
  $= -\langle N_u, X_u\rangle u'^2 - \langle N_u, N_v\rangle u'v' - \langle N_v, X_u\rangle u'v' - \langle N_v, X_v\rangle v'^2$
  $= eu'^2 + 2fu'v' + gv'^2 \quad **$

  where $e = \langle N, X_{uu}\rangle$, $f = \langle N, X_{uv}\rangle = \langle N, X_{vu}\rangle$, and $g = \langle N, X_{vv}\rangle$

$**$ is an equality since $-\langle N, X_u\rangle = -\langle N, X_v\rangle = 0$

↑ ⟨normal vector, tangent vector⟩

$\Rightarrow -\langle N_u, X_u\rangle - \langle N, X_{uu}\rangle = 0$ by taking ∂_u of both sides

$\Rightarrow -\langle N_u, X_u\rangle = \langle N, X_{uu}\rangle = e$

Similarly, $-\langle N_v, X_v\rangle = \langle N, X_{vv}\rangle = g$ and $\overset{0}{\partial_v(-\langle N, X_u\rangle)} = -\langle N_v, X_u\rangle - \langle N, X_{uv}\rangle$

$\Rightarrow -\langle N_v, X_u\rangle = \langle N, X_{uv}\rangle$ // equality here since $\partial_v\partial_u = \partial_u\partial_v$

Similarly, $0 = \partial_u(-\langle N, X_v\rangle) \Rightarrow -\langle N_u, X_v\rangle = \langle N, X_{vu}\rangle = f$

It remains to see how $e, f,$ and g are related to our a_{ij} matrix.

Remember we had

$$N_u = a_{11} X_u + a_{21} X_v$$
$$N_v = a_{12} X_u + a_{22} X_v$$

$$\Rightarrow -e = \langle N_u, X_u \rangle = \langle a_{11} X_u + a_{21} X_v, X_u \rangle$$
$$= a_{11} E + a_{21} F \quad (E, F \text{ from first fundamental form})$$

$$-g = \langle N_u, X_v \rangle = a_{12} F + a_{22} G$$
$$-f = \langle N_u, X_v \rangle = a_{11} F + a_{21} G$$
$$-f = \langle N_v, X_u \rangle = a_{12} E + a_{22} F$$

To make things more clear, let's write this in matrix notation:

$$-\begin{pmatrix} e & f \\ f & g \end{pmatrix} = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} \overbrace{\begin{pmatrix} E & F \\ F & G \end{pmatrix}}^{A}$$

So $\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} = -\begin{pmatrix} e & f \\ f & g \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} = -\begin{pmatrix} e & f \\ f & g \end{pmatrix} \frac{1}{EG - F^2} \begin{pmatrix} G & -F \\ -F & E \end{pmatrix}$

Next time: Why is A invertible?

Asside: Suppose $A = A^T$. let $v = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$. Then $v^T A v = Ex^2 + 2Fxy + Gy^2$

$\Rightarrow$ You can always think of quadratic forms as matrices

Furthermore, this is how you define metrics. For the Euclidean metric, A is the identity matrix.