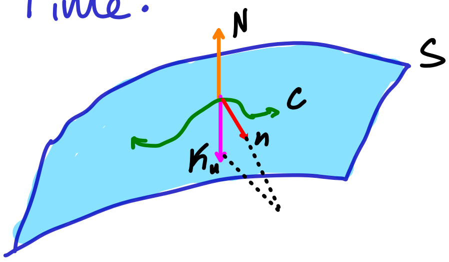
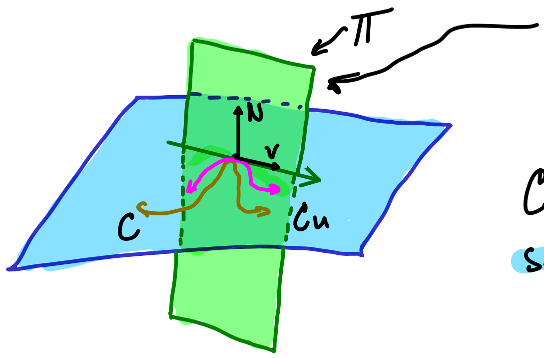
MTH 255 November 4, 2019 Notes

Last Time:



- $k_n = \langle N, \alpha''(0)\rangle$ is called the normal curvature $= II_p(\alpha', 0)$

Proposition: All curves lying on a surface S and having, at a given point $p \in S$, the same tangent line have, at this point, the same normal curvature.



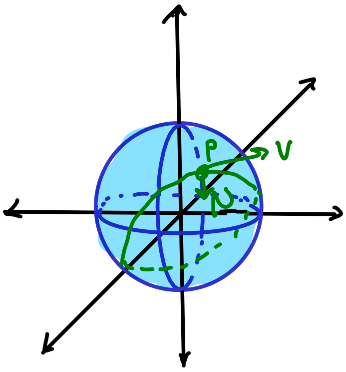
The plane spanned by the tangent line to C at p and N.

$C_n = \pi \cap S$ is called the normal section of S at p along v.

- Note that C_n is a plane curve (since it's contained in π). Thus its normal vector α'' is in π. $\alpha'' \perp v$, so α'' must be parallel to N. Thus, $|k_n| = |\langle N, \alpha''\rangle| = |N||\alpha''| = |\alpha''| = k$. That is, the normal curvature of the normal section C_n is its usual curvature. This implies the normal curvature of a curve C at $p \in S$ whose tangent vector is v is the curvature of the normal section of S along v.

Example: Suppose S is a plane. What can we say about the normal sections of S?
They are just lines because they are the intersections of two planes. The curvature of a line is zero.
$\Rightarrow$ The 2^{nd} fundamental form as well as the differential of the Gauss map is identically zero.

Example: Suppose S is the unit sphere.

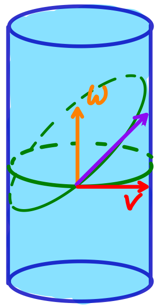


- The normal sections of a sphere are all great circles (radius $= 1$).

  $\Rightarrow$ the curvature is 1.

  $\Rightarrow$ $II_p(v) \equiv 1 \quad \forall\, v$ and $dN_p(v) = v$

Example: Let S be a cylinder.



- The normal section along v is a circle of radius 1 while the normal section along w is a straight line.

  $\Rightarrow$ $II_p(v) = 1$ and $II_p(w) = 0$.

- If $u = av + bw$ where $|v| = |w| = 1$ and $a^2 + b^2 = 1$, then it is possible to find the curvature of the normal section along u geometrically, but we will apply a linear algebra trick.

Recall that $[dN]_{\{v,w\}} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$

$$II_p(u) = \langle dN(u), u \rangle = \left\langle \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} a \\ b \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} a \\ 0 \end{pmatrix}, \begin{pmatrix} a \\ b \end{pmatrix} \right\rangle = a^2 \le 1$$

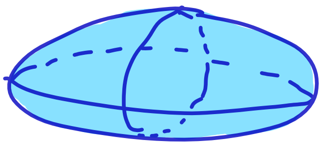
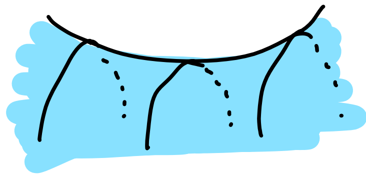
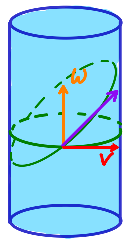
Since $a^2 + b^2 = 1$. So $0 \le II_p(u) \le 1$.

Note that v and w give the extreme values for II_p.

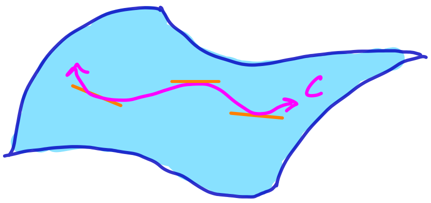
Since dN is self-adjoint, there is an orthonormal basis $\{e_1, e_2\}$ such that $[dN]_{\{e_1, e_2\}}$ is diagonal. Say $dN(e_1) = -k_1 e_1$ and $dN(e_2) = -k_2 e_2$ where $k_1 \ge k_2$. Then $k_2 \le II_p(u) \le k_1$ for any unit vector u.

Definition: The maximum normal curvature k_1 and the minimum normal curvature k_2 are called the principal curvatures at p. The corresponding directions are called principle directions at p.

- In a plane or sphere, all directions are principle.

Definition: Say $\forall p \in C$ the tangent line of C at p is a principle direction at p. Then C is called a line of curvature of S.



Proposition: A necessary + sufficient condition for a curve C to be a line of curvature of S is that

$$N'(t) = \lambda(t)\alpha'(t)$$

where $\alpha(t)$ is a parameter of C, $N(t) = N(\alpha(t))$ and $\lambda : I \to \mathbb{R}$ is a differentiable function. In this case, $-\lambda(t)$ is a principle curvature along $\alpha'(t)$.

Proof:

$$dN(\alpha'(t)) = \frac{d}{dt}N(\alpha(t)) = \frac{d}{dt}N(t) = N'(t) = \lambda(t)\alpha'(t).$$

- $$I(u) = I(av + bw) = -\langle dN\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} a \\ b \end{pmatrix}\rangle$$

$$= -\left\langle \begin{pmatrix} -k_1 & 0 \\ 0 & -k_2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} a \\ b \end{pmatrix} \right\rangle$$

$$= \left\langle \begin{pmatrix} k_1 a \\ k_2 b \end{pmatrix}, \begin{pmatrix} a \\ b \end{pmatrix} \right\rangle = k_1 a^2 + k_2 b^2$$

Definition: Let $p \in S$ and let $dN_p : T_pS \longrightarrow T_pS$ be the differential of the Gauss map. The determinant of dN_p is called the **Gaussian curvature K of S at p**. $-\frac{1}{2}\det(dN_p)$ is called the **mean curvature H of S at p**.

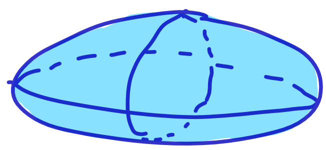
Definition: A point of a surface S is called

1) **Elliptic** if $K_p > 0$
2) **Hyperbolic** if $K_p < 0$
3) **Parabolic** if $K_p = 0$ + $dN_p \neq 0$
4) **Planar** if $K_p = 0$ + $dN_p = 0$

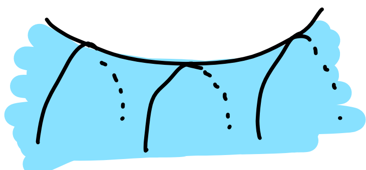
- Note that $\det(N_p) = \det(d\bar{N}_p)$. $\Rightarrow$ The above definition is not dependent on orientation.

Examples:

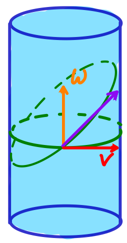
1) S^2 (sphere). Note that any ellipsoid also has positive curvature. Hence the name elliptic.



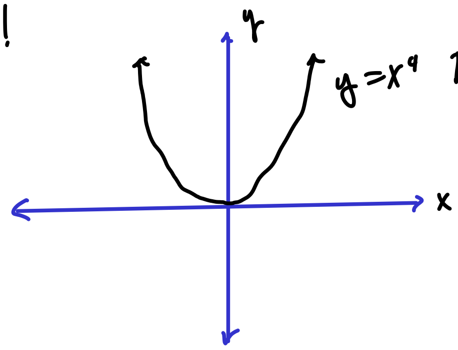
2) $z = y^2 + x^2$ has a hyperbolic point at $(0,0)$.



3) Points on a cylinder: In fact, if you take any parabola + translate it along a line to get a parabolic cylinder has all parabolic points.



4) Of course a plane. However there is a non trivial example here!



$y = x^4$ Rotated around y-axis has a planar point at the origin