MTH 255 October 7, 2019 Notes

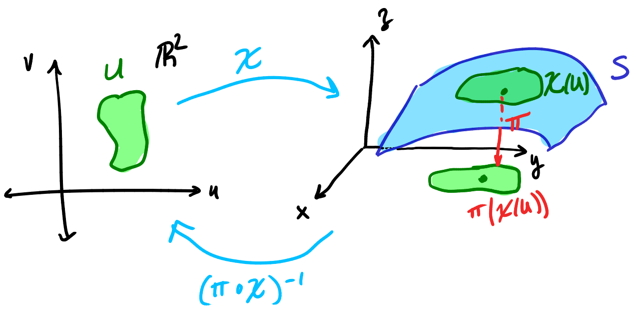
Last Time:

Proposition 1: The graph of $f: U \subset \mathbb{R}^2 \to \mathbb{R}$ is a regular surface

Proposition 3: Given a regular surface S, $p \in S$, $\exists$ a neighborhood V of p in S one of $x = h(y,z)$, $y = f(x,z)$, $z = g(x,y)$ such that V is the graph of one of these functions.

Proposition 4: Let p be a point in a regular surface S and let $\mathbf{x}: U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a map with $p \in \mathbf{x}(U) \subset S$ such that $\mathbf{x}$ is differentiable and $d\mathbf{x}$ is 1-1. Assume that $\mathbf{x}$ is 1-1. Then $\mathbf{x}^{-1}$ is continuous.

Proof:



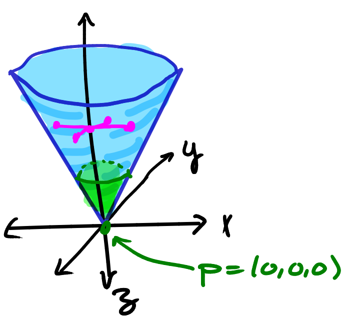
- Without loss of generality, assume

$$0 \neq \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}$$

- $\pi \circ \mathbf{x}$ has an inverse by the inverse function theorem.

- We have $\mathbf{x}^{-1} = \underbrace{(\pi \circ \mathbf{x})^{-1} \circ \pi}_{\text{continuous}}$ $\Rightarrow \mathbf{x}^{-1}$ is also continuous. $\square$

Example: Consider C defined by $z = \sqrt{x^2 + y^2}$ $(x,y) \in \mathbb{R}^2$



- Intuitively, we know that this should not be a regular surface (due to the sharp "point") but how do we prove it?

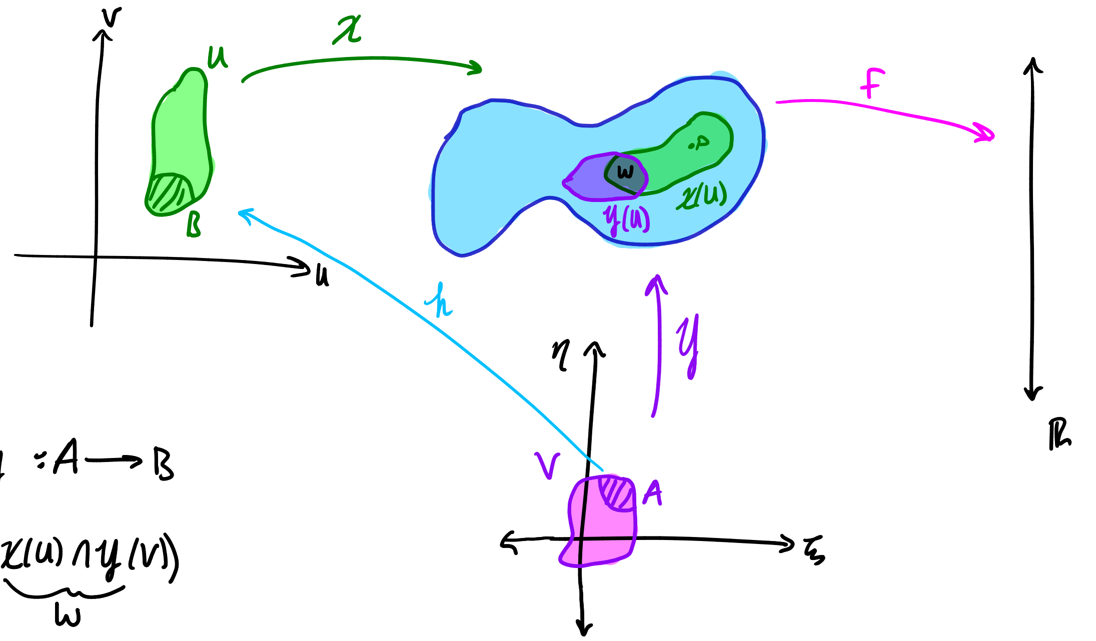
- Use proposition 3. Can C be the graph of $x = h(y,z)$?

- No! Any "functions" describing C of the form $h(y,z)$ or $f(x,z)$ would fail to be functions due to the "vertical line test." (see pink lines & points)

$\Rightarrow$ if C is a regular surface it must be the graph of $z = g(x,y)$.

But $z = \sqrt{x^2 + y^2}$ $\Rightarrow$ This must be a parametrization, but it's not differentiable at $(0,0,0) = p \in C$ $\Rightarrow$ it's not a parametrization. $\Rightarrow$ C is not a regular surface! $\square$

Section 2.2 Change of Parameters: Differentiable functions on surfaces.



Let $h: \mathbf{x}^{-1} \circ \mathbf{y} : A \longrightarrow B$

$$A = \mathbf{x}^{-1}(\underbrace{\mathbf{x}(U) \cap \mathbf{y}(V)}_{W})$$

Proposition 1: Let p be a point of a regular surface S and let $\mathbf{x}: U \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ and $\mathbf{y}: V \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be two parametrizations of S such that $p \in \mathbf{x}(U) \cap \mathbf{y}(V) = W$. Then the "change of coordinates" $h = \mathbf{x}^{-1} \circ \mathbf{y} : A \longrightarrow B$ is a diffeomorphism. That is, h is differentiable and has a differentiable inverse h^{-1}. In other words, if $\mathbf{x}$ & $\mathbf{y}$ are given by

$$\mathbf{x}(u,v) = (x(u,v), y(u,v), z(u,v)) \qquad (u,v) \in U$$

$$\mathbf{y}(\xi,\eta) = (x(\xi,\eta), y(\xi,\eta), z(\xi,\eta)) \qquad (\xi,\eta) \in V$$

the change of coordinates h given by $u = u(\xi,\eta)$ and $v = v(\xi,\eta)$ $(\xi,\eta) \in \mathbf{y}^{-1}(W)$, has the property that u & v have continuous partial derivatives of all orders and the map h can be inverted. $\xi = \xi(u,v)$, $\eta = \eta(u,v)$ $(u,v) \in \mathbf{x}^{-1}(W)$ where ξ, η also have partial derivatives of all orders.

Since $h \circ h^{-1} = id$, $\dfrac{\partial(u,v)}{\partial(\xi,\eta)} \cdot \dfrac{\partial(\xi,\eta)}{\partial(u,v)} = 1$.

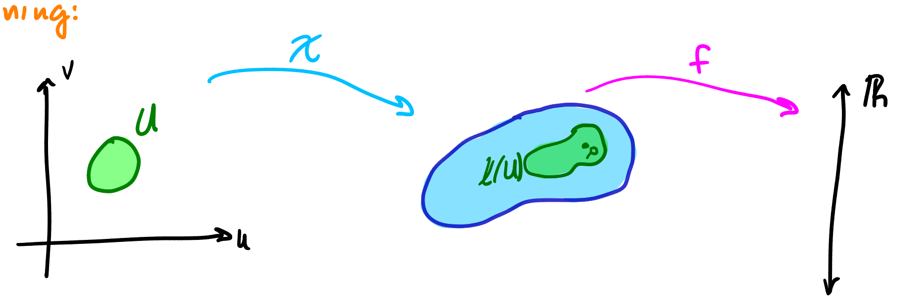
$\Rightarrow$ the Jacobian determinants of both h and h^{-1} have to be nonzero.

Definition: Let $f: V \subset S \longrightarrow \mathbb{R}$ be a function defined in an open subset V of a regular surface S. Then f is said to be **differentiable at $p \in V$** if, for some parametrization $\chi: U \subset \mathbb{R}^2 \longrightarrow S$, with $p \in \chi(U) \subset V$, the composition $f \circ \chi: U \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$ is differentiable at $\chi^{-1}(p)$. f is **differentiable in V** if it is differentiable at all points of V.

- Notice that if $f \circ \chi$ is differentiable then $f \circ y$ is also differentiable because $f \circ y = f \circ \chi \circ \chi^{-1} \circ y$ and $\chi^{-1} \circ y$ is differentiable by proposition 1.

$\Rightarrow$ this definition is coordinate independent.

Warning:

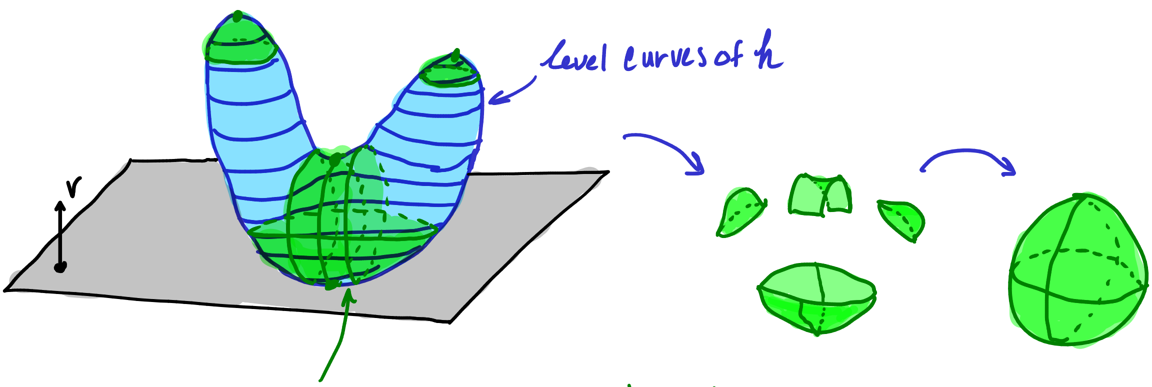


Notation: We identify f, $f \circ \chi$ with $f(u,v)$ and refer to $f(u,v)$ as the **coordinate expression of f.**

We will also identify U with $\chi(U)$.

Example: Let S be a regular surface and $V \subset \mathbb{R}^3$ be an open set such that $S \subset V$. Let $f: V \subset \mathbb{R}^3 \longrightarrow \mathbb{R}$ be differentiable. Then the restriction of f to S (denoted by $f|_S : S \longrightarrow \mathbb{R}$) is a differentiable function on S. In particular,

1. The Height Function relative to a unit vector $v \in \mathbb{R}^3$ $h: S \longrightarrow \mathbb{R}^3$ defined by $h(p) = p \cdot v$. (we normally take $v = e_3$ so that $p \cdot v = z$-coordinate of p)



The critical points of the height function tell you how to "build" surfaces (& other shapes maybe of higher dimension) out of simpler shapes

(This is an area of topology called Morse Theory)