MTH 255 Wednesday, October 30, 2019 Notes

Note about the last homework: If you define a function $f: S \to \mathbb{R}^2$ where S is a surface in $\mathbb{R}^3$. You cannot talk about partial derivatives of f, at least not without being very careful.

A partial derivative is defined like this: $f_x(p_0) = \lim_{h \to 0} \frac{f(p_0 + he_1) - f(p_0)}{h}$

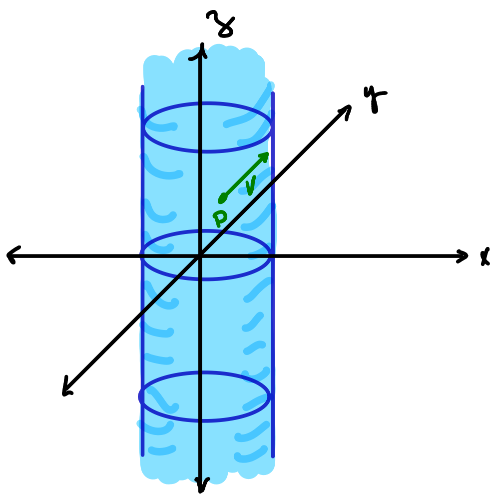
However, this is not defined in general.

If you want to define some function like this: $f(x,y) = (x, y, x^2+y^2)$, there are two possibilities for the domain/range of f.

1. $f: \mathbb{R}^2 \to S$. In this case f is invertible so f^{-1} exists. However, the derivative of f^{-1} does not make sense.
2. $f: \mathbb{R}^2 \to \mathbb{R}^3$. In this case our derivatives are well defined but f is not invertible.

To be safe, use the definition or think of your function as a restriction of a function from $\mathbb{R}^2 \to \mathbb{R}^3$

Example: Consider $S = \{(x,y,z) : x^2+y^2=1\}$



find N and dN. $\exists\, \alpha: I \to S$ where I is any interval (a,b). $\alpha(t_0) = p$ and $\alpha'(t_0) = v$

Say $\alpha(t) = (x(t), y(t), z(t))$.

Then $\alpha'(t_0) = (x'(t_0), y'(t_0), z'(t_0)) = v$

Also $x(t)^2 + y(t)^2 = 1 \quad \forall\, t \in I$ since α is in S.

$2x(t) \cdot x'(t) + 2y(t) y'(t) = 0$ or in other words,

$(x(t), y(t), 0) \cdot (x'(t), y'(t), z'(t)) = 0$

$(x(t_0), y(t_0), 0) \cdot v = 0$

$\uparrow$ arbitrary velocity vector.

So $N(p) = N(x(t_0), y(t_0), z(t_0)) = (x(t_0), y(t_0), 0)$

or $N(x,y,z) = (x,y,0)$ and $\bar{N} = (-x,-y,0)$

Now, what is $dN_p(v)$?

Say $\alpha(t) = (\cos(\theta(t)), \sin(\theta(t)), z)$. $dN_p(v) = dN_p(\alpha'(t_0)) = \frac{d}{dt}(N(\alpha(t)))\Big|_{t=t_0}$

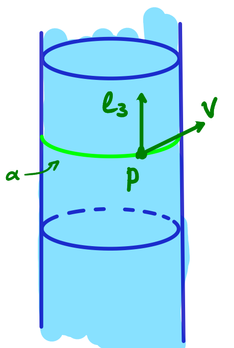
$\alpha'(t_0) = (-\theta'(t)\sin(\theta(t_0)), \theta'(t_0)\cos(t_0), z'(t_0))$

$= \frac{d}{dt} N(t)\Big|_{t=t_0}$ where $N(t) = N(\alpha(t)) = N'(t_0)$

$N(t) = (\cos(\theta(t)), \sin(\theta(t)), 0)$ & $N'(t_0) = \theta'(t)(-\sin(\theta(t_0)), \cos(\theta(t_0)), 0)$

So $dN_p(x,y,z) = (x,y,0)$

- This is <u>not</u> the most efficient way to solve this problem. A better way is to find the eigenvalues of the Gauss map, using geometry.



- $\alpha(t) = p + te_3$ is a curve in S such that $\alpha(0) = p$ and $\alpha'(0) = e_3$ so it represents $e_3 \in T_pS$. where $p = (x,y,z)$
- $N(t) = N(\alpha(t)) = (x,y,0)$. So $N'(t) = (0,0,0)$. In other words, $dN_p(e_3) = 0 = 0 \cdot e_3 \Rightarrow e_3$ is an eigenvector associated to the eigenvalue 0. Now we want to find a second one.

- Let $\alpha(t) = (\cos(t), \sin(t), z)$. Then $\alpha(t_0) = p$ and $\alpha'(t_0) = v$.

$N(t) = N(\alpha(t)) = (\cos(t), \sin(t), 0)$, $N'(t) = (-\sin(t_0), \cos(t_0), 0) = \alpha'(t_0)$

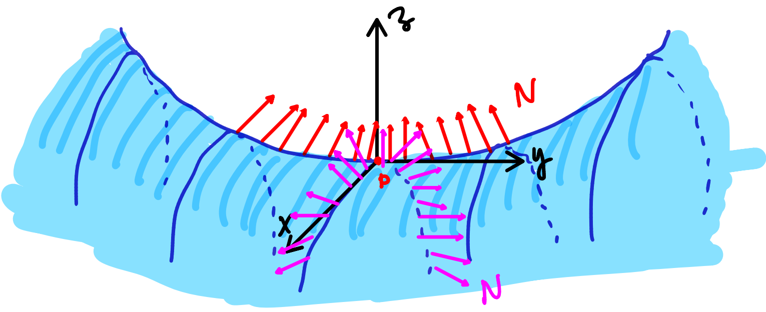
$\Rightarrow dN_p(v) = v = 1 \cdot v \Rightarrow v$ is another eigenvector with eigenvalue 1.

Note: for $\bar{N}$ we have $d\bar{N}(e_3) = 0$ and $d\bar{N}(v) = -v$

In the ordered basis $\{v, e_3\}$ for T_pS, the matrix representation of dN_p is:

$$[dN_p] = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Example: Let $p = (0,0,0)$ inside S where $S = \{(x,y,z) : z = y^2 - x^2\}$. This is called a hyperbolic paraboloid.



- Notice how the red vectors point more towards each other near the origin $\Rightarrow$ the eigenvalue associated to e_2 will be < 0
- On the other hand the pink vectors point more and more away from each other $\Rightarrow$ the eigenvalue associated to e_1 is < 0

$dN_p(e_1) = 2e_1$ and $dN_p(e_2) = -2e_2$

Collecting our results:

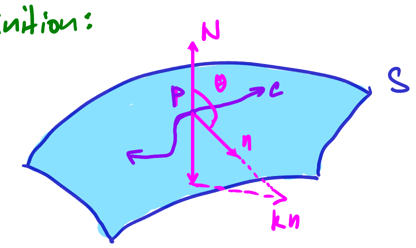
- $[dN_p]_{\{e_1,e_2\}} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$ hyperbolic paraboloid
- $[dN_p] = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ plane
- $[dN_p] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ for the sphere S^2

Proposition: The differential $dN_p : T_pS \longrightarrow T_pS$ of the Gauss map is a self-adjoint linear map.

- A is called self-adjoint if $\langle Av, w\rangle = \langle v, Aw\rangle$. Another characterization is if $\{f_1, f_2\}$ is an orthonormal basis $[A]_{\{f_1,f_2\}}$ is symmetric $(A^T = A)$
  ↑ transpose.

Definition: The quadratic form II_p defined in T_pS by $II_p(v) = -\langle dN_p(v), v\rangle$ is called the second fundamental form of S at p.

Definition:



- k is the curvature of C at p.
- N: unit normal field of S
- n: normal vector of C
- $\cos\theta = \langle N, n\rangle$

We define $k_n = k\cos\theta$. k_n is called the normal curvature of $C \subset S$ at p.

Remark: If $\alpha(s)$ is a parametrization of C by arc length s then

$k_n = \langle N, \alpha''\rangle$

Remark: Changing the orientation of the surface changes the sign of k_n.

- Note that $\langle N(\alpha(s)), \alpha'(s)\rangle = 0$. $\frac{d}{ds}$ of both sides: [Set $N(s) = N(\alpha(s))$].

$$\langle N'(s), \alpha'(s)\rangle + \langle N(s), \alpha''(s)\rangle = 0$$

So $k_n = \langle N(s), \alpha''(s)\rangle = -\langle N'(s), \alpha'(s)\rangle = -\langle dN_{\alpha(s)}\alpha'(s), \alpha'(s)\rangle = II_p(\alpha'(s)) = II_p(v)$

- We see that the normal curvature of a curve C at p only depends on the tangent vector of C at p! In other words, if two curves $\alpha(t), \beta(t)$ have the same tangent vector at p: $\alpha'(t_0) = \beta'(t_0)$, then their normal curvatures are the same.