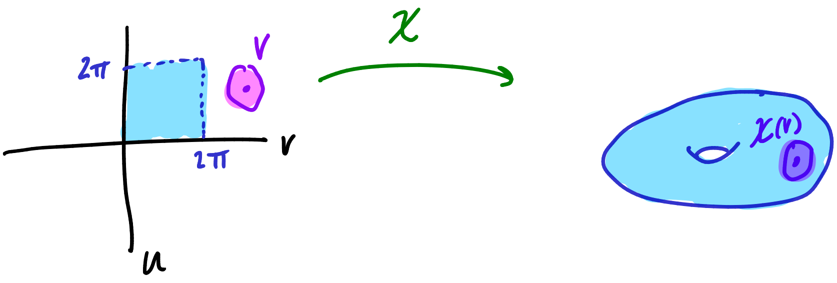
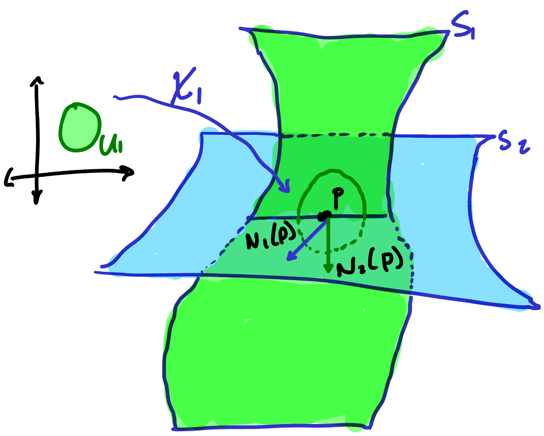
Monday, October 21, 2019 MTH 255 Notes

Last Time: $F: U \subset S_1 \longrightarrow S_2$ is a local diffeomorphism at p if $\exists\, V \subset U$ such that $p \in V$ and $F|_V : V \longrightarrow F(V)$ is a diffeomorphism. F is a local diffeomorphism if it is a local diffeomorphism $\forall\, p \in U$.



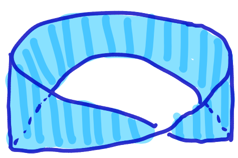
$V \subset U = \mathbb{R}$

Proposition: If $F: U \subset S_1 \longrightarrow S_2$ is a differential mapping of U such that the $dF_p : T_p S_1 \longrightarrow T_p S_2$ is an isomorphism, then F is a local diffeomorphism at p.



- The angle of intersection between S_1 & S_2 at p is defined to be the angle of intersection between $T_p S_1$ and $T_p S_2$.
- $$N_1(p) = \frac{\dfrac{\partial \chi_1}{\partial u} \times \dfrac{\partial \chi_1}{\partial v}}{\left| \dfrac{\partial \chi_1}{\partial u} \times \dfrac{\partial \chi_1}{\partial v} \right|}$$

- A möbius band is not orientable.



Section 2.5 The First Fundamental Form ; Area

- From now on, we will denote the dot product $u \cdot v$ in $\mathbb{R}^3$ by $\langle u,v\rangle$. Given a regular surface $S \subset \mathbb{R}^3$, $p \in S$, and two vectors $v, w \in T_pS$, we denote $v \cdot w$ by $\langle v,w\rangle_p$. ($\langle v,w\rangle_p$ is also called an inner product on T_pS). To this symmetric (you can switch the order of v & w), bilinear (linear in both v and w) form, we can associate a quadratic form I_p defined by $I_p(w) = \langle w,w\rangle_p$ (which is like norm squared, $|w|^2$)
- Over $\mathbb{R}$, symmetric bilinear forms are in one-to-one correspondence with quadratic forms: If you have $B(v,w)$ we can define $Q(w) = B(w,w)$. Conversely, if you have $Q(w)$ we can define $B(v,w) = \frac{1}{2}\left(Q(v+w) - Q(v) - Q(w)\right)$

Definition: The quadratic form I_p on T_pS is called the First Fundamental Form of $S \subset \mathbb{R}^3$ at p.

- In local coordinates $X: U \to S$, given $w \in T_pS$ ($p \in U$) then $w = aX_u + bX_v$ for some $a, b \in \mathbb{R}$. Then

$$\begin{aligned} I_p(w) = \langle w,w\rangle_p &= \langle aX_u + bX_v, aX_u + bX_v\rangle_p \\ &= a^2\langle X_u, X_u\rangle_p + 2ab\langle X_u, X_v\rangle + b^2\langle X_v, X_v\rangle_p \\ &= a^2|X_u|_p^2 + 2ab\langle X_u, X_v\rangle_p + b^2|X_v|_p^2 \\ &= a^2E + 2abF + b^2G \end{aligned}$$

where $E_p = |X_u|_p^2$, $F_p = \langle X_u, X_v\rangle_p$, $G_p = |X_v|_p^2$. So $E, F, G: U \to \mathbb{R}$ describe the local geometry. Warning: E_p, F_p, G_p are not geometric invariants, although their behavior under coordinate transformations can be described.

Example: Let $P \subset \mathbb{R}^3$ be the plane that passes through p_0 containing the orthonormal vectors w_1 and w_2. Then a parametrization for P is given by $X(u,v) = p_0 + uw_1 + vw_2$. So $E = |X_u|^2 = |w_1|^2 = 1$. Similarly, $F = \langle X_u, X_v\rangle = \langle w_1, w_2\rangle = 0$ and $G = |X_v|^2 = |w_2|^2 = 1$.

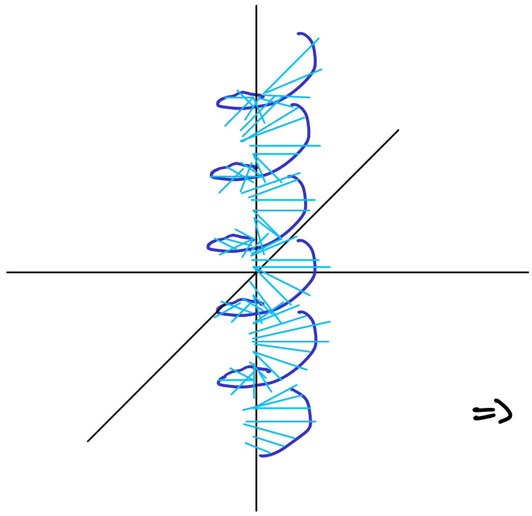
If $w = a\mathcal{X}_u + b\mathcal{X}_v$ is a tangent vector at $\mathcal{X}(p)$, then

$$I_p(w) = a^2 E_p + 2ab F_p + b^2 G_p = a^2 + b^2$$

Example: Let $C \subset \mathbb{R}^3$ be the right cylinder parametrized by $\mathcal{X}(u,v) = (\cos u, \sin u, v)$ for $0 < u < 2\pi$, $v \in \mathbb{R}$. Then $\mathcal{X}_u = (-\sin u, \cos u, 0)$, $\mathcal{X}_v = (0,0,1)$, so we have:

$$E = 1 \qquad F = 0 \qquad G = 1$$

Example (Helicoid):



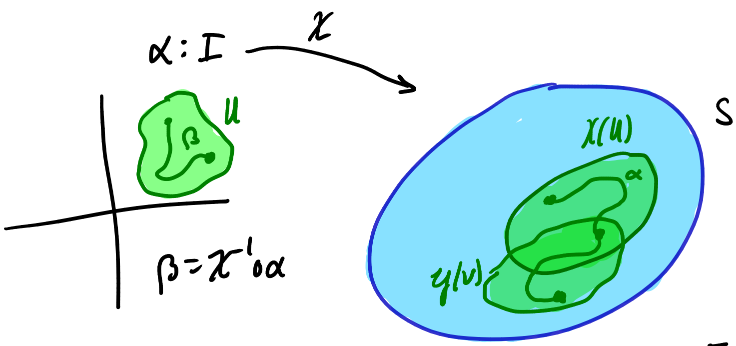
The helix is given by $\alpha(t) = (\cos(t), \sin(t), at)$

$$\mathcal{X}(t,u) = (u\cos t, u\sin t, at)$$
$$= (0,0,at) + u(\cos t, \sin t, 0)$$

$$\mathcal{X}_t = (-u\sin t, u\cos t, a)$$
$$\mathcal{X}_u = (\cos t, \sin t, 0)$$

$$\Rightarrow \quad E = u^2\sin^2 t + u^2\cos^2 t + a^2 = u^2 + a^2$$
$$F = -u\cos t\sin t + u\cos t\sin t + 0 = 0$$
$$G = 1$$

Interesting Note: You cannot find a parametrization for the helicoid such that E, F, and G are all constant.



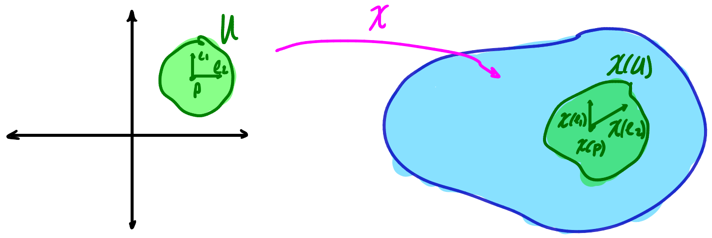
- If we have the situation at the left, then the arc length of β is given by

$$\int_0^s \sqrt{|\beta'(t)|^2}\,dt = \int_0^s \sqrt{|\mathcal{X}_u u'(t) + \mathcal{X}_v v'(t)|^2}\,dt$$
$$= \int_0^s \sqrt{(u')^2 E + 2u'v' F + (v')^2 G}\,dt$$

Note: To take care of the whole curve we might need more than one parametrization.

If $v, w \in \mathbb{R}^3$ the angle between them is given by $\cos\theta = \frac{\langle v, w \rangle}{|v||w|}$.
In particular, for $v = X_u$, $w = X_v$ we have

$$\cos\theta = \frac{\langle X_u, X_v \rangle}{|X_u||X_v|} = \frac{F}{\sqrt{EG}}$$



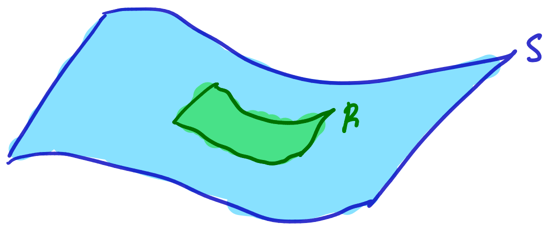
- So if $F \equiv 0$ then X_u, X_v are perpendicular at every point $p \in U$. Such parametrizations are called orthogonal.

Area

Definition: A regular domain of S is an open and connected subset of S such that its boundary is the image of a circle by a differentiable homeomorphism which is regular except at finitely many points.

Definition: A region is a regular domain union its boundary.

Definition: A region of $S \subset \mathbb{R}^3$ is bounded if it is contained in some ball in $\mathbb{R}^3$.



- Here R is a bounded region
- For example the upper ½-plane in $\mathbb{R}^3$; $\{(x, y, z) : x \geq 0, z = 0\}$ is an unbounded region in the surface in $\mathbb{R}^3$ which is the x-y plane.