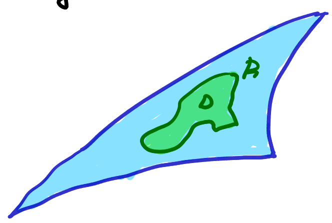
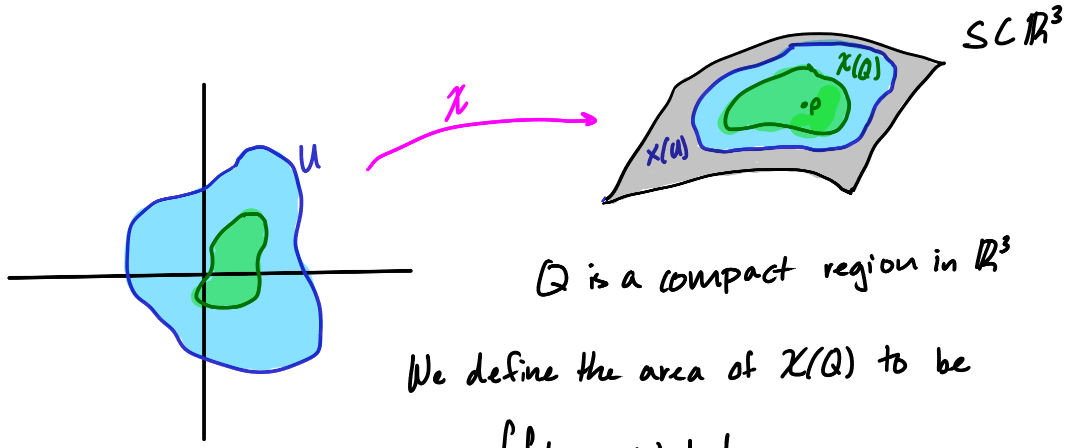
MTH 255 Monday October 25, 2019 Notes

Last time: $D \subset S$ is called a regular domain if it is an open and connected subset and its boundary is a "piecewise C^∞ circle". $R \subset S$ is a region if

$R = D \cup \underset{\uparrow \text{boundary}}{\partial D}$ where D is a domain.





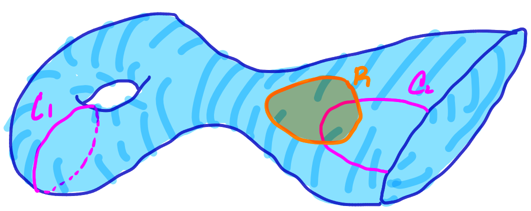
Q is a compact region in $\mathbb{R}^3$

We define the area of $X(Q)$ to be

$$\iint_Q |X_u \times X_v|\, du\, dv$$

- Recall $|X_u \times X_v|$ is the area of a parallelogram spanned by X_u and X_v.
- If you choose a different parametrization $y: V \longrightarrow S^2$ such that $y(Q') = X(Q)$, $y(\bar{u},\bar{v}) \in S^2$

then $\iint_{Q'} |y_{\bar{u}} \times y_{\bar{v}}|\, d\bar{u}\, d\bar{v}$ has the same value by a change of variables for multiple integrals.



- You can always find a parametrization X of S such that $S \setminus \operatorname{Im} X = C_1 \cup C_2 \cup \cdots \cup C_k$ where C_i are curves.

Recall that $|X_u \times X_v|^2 + |\underbrace{X_u \cdot X_v}_{F}|^2 = \underbrace{|X_u|^2}_{E}\underbrace{|X_v|^2}_{G}$ because $|u \times v|^2 = \det\begin{pmatrix} u\cdot u & u \cdot v \\ v \cdot u & v \cdot v \end{pmatrix}$

$\Rightarrow |X_u \times X_v| = \sqrt{EG - F}$ $\Rightarrow$ Area of $X(Q) = \iint_Q \sqrt{EG - F^2}\, du\, dv$

Homework Problem: Find the surface area of S^2 using spherical coordinates and following Example 5 in section 2.5.

Section 3.2: The Gauss map and its fundamental properties.

Definition: If $V \subset S$ is an open set in S and $N: V \to \mathbb{R}^3$ is a differentiable map that associates to each point $p \in V$ a unit normal vector at p, we say that N is a differentiable field of unit normal vectors.

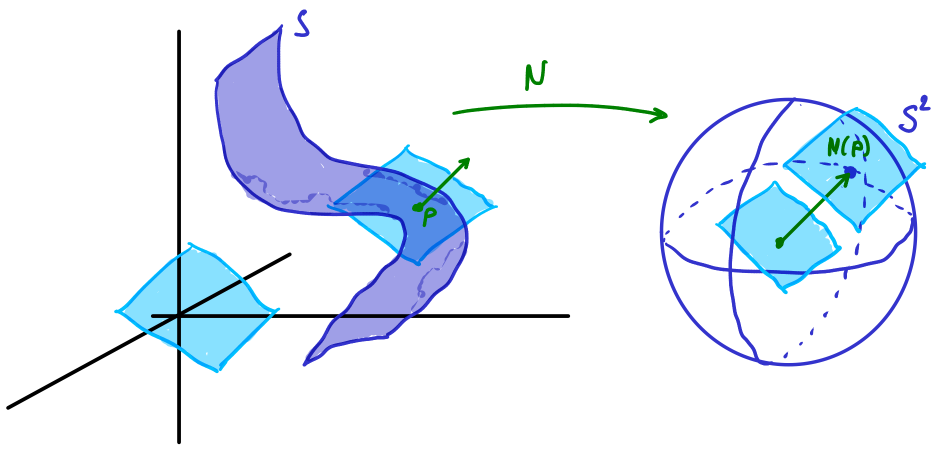
Locally, we can always find a field of unit normal vectors since near every point we have a parametrization X, we may set $N: U \to \mathbb{R}^3$ to be $N = \dfrac{X_u \times X_v}{|X_u \times X_v|}$

Definition: A surface S is called orientable if there exists a field of unit normal vectors $N: S \to \mathbb{R}^3$. Note that $\bar{N}$ defined by $\bar{N}(p) = -N(p)$ is another choice of a unit normal vector field. => If S is orientable, there are only two possible orientations.* A surface together with a choice of N or $\bar{N}$ is called an oriented surface. S is called non-orientable if there is no global unit normal vector field.

* Of course, there are only two choices per connected component of S.

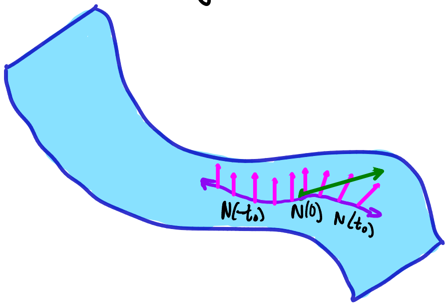
Let $\{v, w\}$ be a basis for T_pS. If $\langle v \times w, N \rangle$ is positive we say $\{v, w\}$ is positively oriented.

Definition: Let $S \subset \mathbb{R}^3$ be a surface with an orientation N. The map $N: S \to \mathbb{R}^3$ takes values in the unit sphere S^2 so $N: S \to S^2$. This map is called the Gauss map. Note that the Gauss map is only defined for orientable surfaces.



So actually $dN_p: T_pS \to T_{N(p)}S^2$ can be written as $dN_p: T_pS \to T_pS$

Question: What does dN_p measure? What is $dN_p(v)$ for $v \in T_pS$?
Represent v by a curve α. ($\alpha(0) = p$, $\alpha'(0) = v$)



- $dN_p(v) = dN_p(\alpha'(0))$

$$= \frac{d}{dt} N(\alpha(t))\Big|_{t=0} = \frac{d}{dt} N(t)\Big|_{t=0} = N'(0)$$

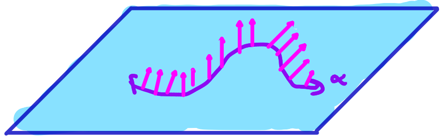
- $N(t) \leftarrow$ notation

$= N(\alpha(t))$ is the normal field N restricted to the curve.

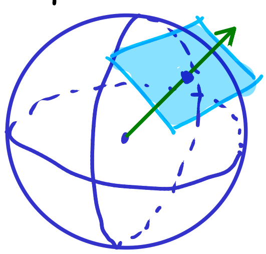
Examples: For a plane $ax+by+cz+d=0$.

$$N = N(x,y,z) = \frac{(a,b,c)}{\sqrt{a^2+b^2+c^2}} \qquad N(\alpha(t)) = \frac{(a,b,c)}{\sqrt{a^2+b^2+c^2}} \quad \forall \text{ curves } \alpha$$

$\Rightarrow dN = 0$ since N is a constant vector field.



Example: $S^2 = \{(x,y,z) : x^2+y^2+z^2 = 1\}$. Let $\alpha(t) = (x(t), y(t), z(t)) \in S^2$
so $x^2(t) + y^2(t) + z^2(t) = 1$ $\forall t$. Taking derivative of both sides we get
$2x(t)x'(t) + 2y(t)y'(t) + 2z(t)z'(t) = 0$ So at $t=0$, we get
$(x(0), y(0), z(0)) \cdot (x'(0), y'(0), z'(0)) = 0$. Since α represents an arbitrary
$\alpha'(0) \in T_{\alpha(0)}S^2$ $\alpha(0)$ is actually a normal vector at $\alpha(0)$. So we may choose
$N(p) = p$ or $N(p) = -p$. For $p \in S^2$ what is dN?

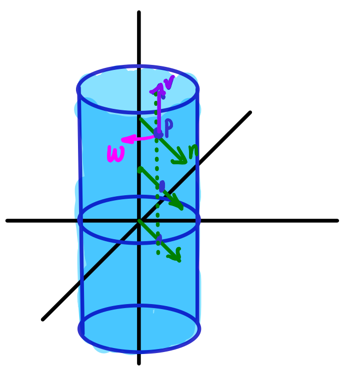


Recall that d(Linear map) = itself.

So $dN_p(v) = v$ (also $d\bar{N}(v) = -v$)

Example: Let $S = \{(x,y,z) : x^2+y^2 = 1\}$

$N = N(x,y,z) = (x,y,0)$



$dN(v) = 0 = 0v$

$dN(w) = w = 1 \cdot w$

$\Rightarrow$ Eigenvalues of dN are $0, 1$

represent w by $\alpha(t) = (\cos(t), \sin(t), z_0)$

$w = \alpha'(t_0) \in T_{\alpha(t_0)}S$

- More about this later.