Midterm - Solutions

Üstün Yıldırım

October 30, 2019

1. See the solution of the Problem 1 in Exercise Set 1: `https://ustunyildirim.com/wiki/lib/exe/fetch.php?media=teaching:fall_2019_mth_255_differential_geometry:homework_and_exercise:exerciseset1_solutions.pdf`
2. (a) See Definition 1 in Section 2.2.
   (b) See Definition 2 in Section 2.2.
3. This is a slight variation of Problem 4 in HW Set 3. For completeness, we write the solution here.
   (a) Recall that $f(x, y, z) = (z - x^2 - y^2)^2$.
   $$\begin{aligned} f_x &= 2(z - x^2 - y^2)(-2x) \\ f_y &= 2(z - x^2 - y^2)(-2y) \\ f_z &= 2(z - x^2 - y^2) \end{aligned}$$
   They are all equal to zero simultaneously if and only if $z - x^2 - y^2 = 0$. So, the critical points are given by $\{(x, y, z) \in \mathbb{R}^3 \mid z = x^2 + y^2\}$. Given a critical point p, we clearly have $f(p) = 0$. Thus, 0 is the only critical value.
   (b) By Proposition 2, $f^{-1}(c)$ is a regular surface for any $c \neq 0$. Also,
   $$\begin{aligned} f^{-1}(0) &= \{(x, y, z) \mid (z - x^2 - y^2)^2 = 0\} \\ &= \{(x, y, z) \mid (z - x^2 - y^2) = 0\} \end{aligned}$$
   which is a paraboloid. To see that it is a regular surface we may use $f(x, y) = x^2 + y^2$ in Proposition 1.
   (c) Recall that $g(x, y, z) = z^2 - y^2$.
   $$(g_x, g_y, g_z) = (0, -2y, 2z)$$
   So, the critical points are $\{(x, 0, 0) \mid x \in \mathbb{R}\}$. So, the only critical value is 0.
   (d) Therefore, $f^{-1}(c)$ is a regular surface for $c \neq 0$.

   Note that $f^{-1}(0)$ is the union of two planes $z = y$ and $z = -y$. If this union was a regular surface, then by Proposition 3 we could represent it, near the origin, as the graph of a function that has one of the following three forms $z = f(x, y), y = g(x, z)$ or $x = h(y, z)$. However, this union fails "the vertical line test" for all three forms. Therefore, it is not a regular surface.[1]

[1]In this problem we saw that the inverse image of a critical point may or may not be a regular surface.

4. See the solution of Problem 2 in HW Set 3: `https://ustunyildirim.com/wiki/lib/exe/fetch.php?media=teaching:fall_2019_mth_255_differential_geometry:homework_and_exercise:hw3_sols.pdf`

5. We solve both parts simultaneously. A plane curve is given by $\alpha(s) = r(\cos(\theta), \sin(\theta))$ where r and θ are functions of s. Clearly, the magnitude of $\alpha(s)$ is r but we are given that $|\alpha(s)| = \frac{s}{c}$. So, we have $r(s) = \frac{s}{c}$. (Technically, $r(s) = -\frac{s}{c}$ is also possible, but this only corresponds to a rotation by π which we will allow later in the solution anyway. So, $r(s) > 0$ is a reasonable assumption.)

   Next, we need to make sure α is parametrized by arc length s. So we compute

$$\begin{aligned} 1 &= |\alpha'(s)|^2 \\ &= \left(r'(\cos(\theta), \sin(\theta)) + r\theta'(-\sin(\theta), \cos(\theta))\right)^2 \\ &= r'^2 + (r\theta')^2 \\ &= \frac{1}{c^2} + \frac{s^2\theta'^2}{c^2}. \end{aligned}$$

   Thus,

$$\theta' = \pm\frac{\sqrt{c^2-1}}{s}$$

   and

$$\theta = \pm\sqrt{c^2-1}\ln(s) + k$$

   for some constant k. So, we get

$$\alpha(s) = \frac{s}{c}\left(\cos\left(\pm\sqrt{c^2-1}\ln(s) + k\right), \sin\left(\pm\sqrt{c^2-1}\ln(s) + k\right)\right).$$

   Note that adding π to k rotates the curve by π about the origin.