MTH 255 - Midterm Exam

October 23, 2019

Justify all your answers.

1. (8 points) Let $\alpha : \mathbb{R} \to \mathbb{R}^3$ be a parametrized differentiable curve such that $\alpha''(t) = 0$ for all $t \in \mathbb{R}$. What geometric shapes can the trace of α be?

2. Your statements in this question should be precise. If you are talking about a point, state where it belongs to (e.g. $p \in S, x \in \mathbb{R}^2$, etc.). If you are talking about a function, state its domain, range and properties (including properties of the domain and range if there is anything specific about them) (e.g. "Let S be a regular surface, $U \subset \mathbb{R}^{255}$ be closed and $F : U \to S$ be continuous. . . ").

   (a) (4 points) State the definition of a regular surface.

   (b) (4 points) State the definition of a critical point, critical value, and regular value for functions from a subset of $\mathbb{R}^n$ to $\mathbb{R}^k$.

3. (a) (4 points) Let $f(x, y, z) = (z - x^2 - y^2)^2$. Find the critical points and critical values of f.

   (b) (2 points) For what values of c is the set $f(x, y, z) = c$ a regular surface?

   (c) (4 points) Let $g(x, y, z) = z^2 - y^2$. Find the critical points and critical values of g.

   (d) (2 points) For what values of c is the set $g(x, y, z) = c$ a regular surface?

4. (10 points) Let T denote the torus defined by

$$T = \left\{ (x, y, z) \in \mathbb{R}^3 \mid \left(\sqrt{x^2 + y^2} - a \right)^2 + z^2 = r^2 \right\}$$

   for $r < a$. Given that T is a regular surface, show

$$X(u, v) = ((r \cos u + a) \cos v, (r \cos u + a) \sin v, r \sin u)$$

   for $0 < u, v < 2\pi$, is a parametrization for T. Hint: you may use the fact that if $(\cos(t), \sin(t)) = (\cos(t'), \sin(t'))$ for $0 < t, t' < 2\pi$, then $t = t'$.

5. (a) (10 points) Find a differentiable plane curve $\alpha : (0, \infty) \to \mathbb{R}^2$ parametrized by arc length s satisfying

$$|\alpha(s)| = \frac{s}{c}$$

   where $c > 1$. (Write an explicit expression for $\alpha(s)$ in terms of s.)

   (b) (2 points) Find all differentiable plane curves satisfying the conditions in part (a).