Homework 4 - Solutions

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1. Clearly the function $f : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $f(x, y, z) = (-x, -y, -z)$ is differentiable. Therefore, its restriction to any surface is a differentiable map. Clearly, it restricts to be the antipodal map A on S^2. Therefore, A is differentiable. Note that $f(f(x, y, z)) = (x, y, z)$. Thus, f is invertible and its inverse is itself. Hence, A is also invertible and its inverse is itself which is a differentiable map. Thus, A is a diffeomorphism.

2. Denote the paraboloid $z = x^2 + y^2$ by S_1 and the xy-plane ($z = 0$) by S_2. Define $f, g : \mathbb{R}^3 \to \mathbb{R}^3$ by $f(x, y, z) = (x, y, 0)$ and $g(x, y, z) = (x, y, x^2 + y^2)$. Clearly, they are both differentiable functions. So, their restrictions $f|_{S_1}$ and $g|_{S_2}$ are also differentiable. It is also clear that the codomain of $f|_{S_1}$ is S_2 and the codomain of $g|_{S_2}$ is S_1. Now, it is easy to verify that $g(f(x, y, x^2 + y^2)) = g(x, y, 0) = (x, y, x^2 + y^2)$ and $f(g(x, y, 0)) = f(x, y, x^2 + y^2) = (x, y, 0)$. So, $f : S_1 \to S_2$ is invertible and its inverse $g : S_2 \to S_1$ is also differentiable. In other words, f is a diffeomorphism from S_1 to S_2. Thus, S_1 and S_2 are diffeomorphic.

3. A relation $R(a, b)$ is an equivalence relation if it satisfies the following three properties: reflexive (that is, every object is related to itself or $R(a, a)$ for all a), symmetric (that is, if a is related to b then b is related to a or if $R(a, b)$ then $R(b, a)$) and transitive (that is, if a is related to b and b is related to c then a is related to c or if $R(a, b)$ and $R(b, c)$ then $R(a, c)$). For us, $R(S_1, S_2)$ represents "S_1 is diffeomorphic to S_2".

   First, we prove reflexive property. Clearly, $f(x, y, z) = (x, y, z)$ is a differentiable function from $\mathbb{R}^3 \to \mathbb{R}^3$. Thus, it restricts to any surface $f : S \to S$ as a differentiable map. It is clearly invertible and its inverse is itself. Therefore, $f : S \to S$ is a diffeomorphism. In other words, any surface is diffeomorphic to itself.

   Next, we prove symmetric property. Let S be diffeomorphic to S'. Then, there is a diffeomorphism $f : S \to S'$. Now, since f is a diffeomorphism, it has an inverse $f^{-1} : S' \to S$ which is differentiable. Since $(f^{-1})^{-1} = f$, f^{-1} is also invertible and its inverse f is differentiable. Thus, $f^{-1} : S' \to S$ is a diffeomorphism and S' is diffeomorphic to S.

   Finally, we prove transitive property. Let S be diffeomorphic to S' and S' be diffeomorphic to S''. Then, we have diffeomorphisms $f : S \to S'$ and $g : S' \to S''$. Since composition of differentiable maps is a differentiable map, $g \circ f : S \to S''$ is differentiable. Note that $(f^{-1} \circ g^{-1}) \circ (g \circ f) = f^{-1} \circ (g^{-1} \circ g) \circ f = f^{-1} \circ (Id_{S'}) \circ f = f^{-1} \circ f = Id_S$. Similarly, $(g \circ f) \circ (f^{-1} \circ g^{-1}) = g \circ (f \circ f^{-1}) \circ g^{-1} = g \circ (Id_{S'}) \circ g^{-1} = g \circ g^{-1} = Id_S$. Thus, $g \circ f$ is invertible and the inverse is $f^{-1} \circ g^{-1}$. Since f^{-1} and g^{-1} are differentiable, the composition is also differentiable and thus, $g \circ f : S \to S''$ is a diffeomorphism. Hence, S is diffeomorphic to S''.

4. Using the notation from the hint, we have $f(\alpha(t)) = 0$. Taking derivative of both sides and

evaluating at $t = 0$, we get

$$\begin{aligned} 0 &= (f_x(\alpha(0)), f_y(\alpha(0)), f_z(\alpha(0))) \cdot \alpha'(0) \\ &= f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0). \end{aligned}$$

5. Set $f(x, y, z) = x^2 + y^2 - z^2 - 1$. Then, $f_x = 2x$, $f_y = 2y$ and $f_z = 2z$. So, the only critical point is the origin. Since $f(0, 0, 0) = -1$, the only critival value is -1. In particular, $f^{-1}(0)$ is a regular surface. By the previous problem, the tangent plane at $(x_0, y_0, 0)$ is given by

$$\begin{aligned} &f_x(x_0, y_0, 0)(x - x_0) + f_y(x_0, y_0, 0)(y - y_0) + f_z(x_0, y_0, 0)(z) \\ &= 2x_0(x - x_0) + 2y_0(y - y_0) = 0. \end{aligned}$$

Clearly, the normal vector is given by $N = (2x_0, 2y_0, 0)$. Since $N \cdot e_3 = 0$, the plane is parallel to the z-axis.