Homework 3 - Solutions

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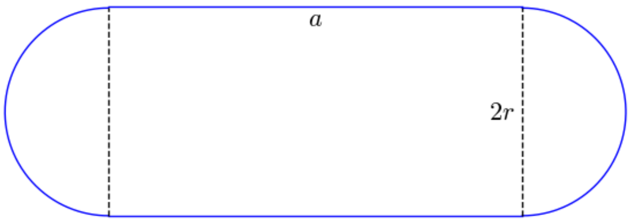
1. (a) By the isoperimetric inequality, any such curve has to satisfy $4\pi A \leq l^2$. However, the given parameters ($A = 12$ and $l = 12$) clearly violate the inequality therefore, such a curve does not exists.

   (b) To construct this curve, we "glue" two semicircles with straight line segments as in the figure below.



If the length of each of the line segments is a and the radius of each of the semicircles is r, the perimeter and the area of this curve is given by

$$l = 2a + 2\pi r$$
$$A = 2ar + \pi r^2.$$

Now, for $a = 2\sqrt{3}\pi$ and $r = 4 - 2\sqrt{3}$, we have $l = 8\pi, A = 4\pi$. So, we all have to show is this curve admits a regular C^1 parametrization. The parametrization can be given as

$$\alpha(t) = \begin{cases} r(\cos(t), \sin(t)) + (a/2, 0) & \text{for } t \in [-\pi/2, \pi/2) \\ (a/2 - r(t - \pi/2), r) & \text{for } t \in [\pi/2, \pi/2 + a/r) \\ r(\cos(t - a/r), \sin(t - a/r)) - (a/2, 0) & \text{for } t \in [\pi/2 + a/r, 3\pi/2 + a/r) \\ (-a/2 + r(t - (3\pi/2 + a/r)), -r) & \text{for } t \in [3\pi/2 + a/r, 3\pi/2 + 2a/r] \end{cases}$$

Now, it is straigth forward to check that α is continuous, and $\alpha(-\pi/2) = \alpha(3\pi/2 + 2a)$. Moreover, it is differentiable and the derivative is

$$\alpha(t) = \begin{cases} r(-\sin(t), \cos(t)) & \text{for } t \in [-\pi/2, \pi/2) \\ (-r, 0) & \text{for } t \in [\pi/2, \pi/2 + a/r) \\ r(-\sin(t - a/r), \cos(t - a/r)) & \text{for } t \in [\pi/2 + a/r, 3\pi/2 + a/r) \\ (r, 0) & \text{for } t \in [3\pi/2 + a/r, 3\pi/2 + 2a/r] \end{cases}$$

which is also continuous and clearly never vanishes.

2. Since sums and products of smooth (C^∞) functions is smooth, X is smooth (differentiable).

Next, we check if the differential is 1-1. Set $X(u,v) = (x(u,v), y(u,v), z(u,v))$. The partial derivatives X_u and X_v are given by

$$\begin{aligned} X_u &= (-r\sin u\cos v, -r\sin u\sin v, r\cos u) \\ X_v &= (-(r\cos u + a)\sin v, (r\cos u + a)\cos v, 0)\,. \end{aligned}$$

So, the 2x2 minor $\frac{\partial(x,y)}{\partial(u,v)}$ is

$$\begin{aligned} \frac{\partial(x,y)}{\partial(u,v)} &= -r\sin u(r\cos u + a)\cos^2 v - r\sin u(r\cos u + a)\sin^2 v \\ &= -r\sin u(r\cos u + a)(\cos^2 v + \sin^2 v) \\ &= -r\sin u(r\cos u + a). \end{aligned}$$

If this minor is not 0, X_u and X_v are linearly independent. So, we may assume that it is equal to zero. However, since $a > r > 0$ and $-1 \le \cos u \le 1$, $r\cos u + a$ is always positive. So, if the minor is 0, then we have $\sin u = 0$. In that case $\cos u = \pm 1$ and

$$\begin{aligned} X_u &= (0, 0, \pm r) \\ X_v &= (-(\pm r + a)\sin v, (\pm r + a)\cos v, 0) \\ &= (\pm r + a)(-\sin v, \cos v, 0). \end{aligned}$$

So, X_u and X_v are clearly linearly independent. In other words, dX is 1-1.

Next, we show that $X(u,v) \subset T$.

$$\begin{aligned} \left(\sqrt{x^2+y^2} - a\right)^2 + z^2 &= \left(\sqrt{(r\cos u + a)^2\cos^2 v + (r\cos u + a)^2\sin^2 v} - a\right)^2 + (r\sin u)^2 \\ &= \left(\sqrt{(r\cos u + a)^2(\cos^2 v + \sin^2 v)} - a\right)^2 + (r\sin u)^2 \\ &= \left(\sqrt{(r\cos u + a)^2} - a\right)^2 + (r\sin u)^2 \\ &= ((r\cos u + a) - a)^2 + (r\sin u)^2 \\ &= (r\cos u)^2 + (r\sin u)^2 \\ &= r^2. \end{aligned}$$

Finally, we show X is 1-1. This is the last thing we need to show since then Proposition 4, tells us that X is a homeomorphism. Assume we have $X(u,v) = X(u',v')$. The first two components of this equality gives us

$$(r\cos u + a)(\cos v, \sin v) = (r\cos u' + a)(\cos v', \sin v').$$

These expressions are both of the form cw where c is a positive constant and w is a unit vector. So, comparing the magnitudes, we see the constants must be the same. Then, dividing by the common positive constant, we see the two unit vectors are the same. Of course, there is a unique angle between 0 and 2π that represents a unit vector. Thus, $v = v'$. Comparing the positive constants again we also see that $\cos u = \cos u'$. Comparing the third component of $X(u,v) = X(u',v')$ we also have $\sin u = \sin u'$. So, again since $0 < u, u' < 2\pi$, and $(\cos u, \sin u) = (\cos u', \sin u')$, we also have $u = u'$ which was to be shown.

3. (a) Let $U = \{(x, y) \mid x^2 + y^2 > 1\}$ and $f : U \to \mathbb{R}$ be defined by

$$f(x, y) = \sqrt{x^2 + y^2 - 1}.$$

Clearly, f is differentiable on U. Also, it is easy to see that the graph of f is a subset of the hyperboloid of one sheet. The fact that $X(x, y) = (x, y, f(x, y))$ is a parametrization now follows from the proof of Proposition 1 (Section 2.2).

(b) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x, y) = \sqrt{x^2 + y^2 + 1}.$$

Clearly, f is differentiable. Also, it is easy to see that the graph of f is a subset of the hyperboloid of two sheets. The fact that $X(x, y) = (x, y, f(x, y))$ is a parametrization now follows from the proof of Proposition 1 (Section 2.2).

4. (a)

$$f_x = f_y = f_z = 2(x + y + z - 1)$$

So, the critical points are given by $\{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 1\}$. Given a critical point p, we clearly have $f(p) = 0$. Thus, 0 is the only critical point.

(b) By Proposition 2, $f^{-1}(c)$ is a regular surface for any $c \neq 0$. Also,

$$\begin{aligned} f^{-1}(0) &= \{(x, y, z) \mid (x + y + z - 1)^2 = 0\} \\ &= \{(x, y, z) \mid (x + y + z - 1) = 0\} \end{aligned}$$

which is a plane. To see that it is a regular surface we may use $f(x, y) = 1 - x - y$ in Proposition 1.

(c)

$$(f_x, f_y, f_z) = (yz^2, xz^2, 2xyz)$$

So, the critical points are $\{(0, 0, z) \mid z \in \mathbb{R}\} \cup \{(x, y, 0) \mid x, y \in \mathbb{R}\}$. So, the only critical value is 0. Therefore, $f^{-1}(c)$ is a regular surface for $c \neq 0$.

Note that $f^{-1}(0)$ is the union of the three coordinate planes. If this union was a regular surface, then by Proposition 3 we could represent it, near the origin, as the graph of a function that has one of the following three forms $z = f(x, y), y = g(x, z)$ or $x = h(y, z)$. However, this union fails "the vertical line test" for all three forms. Therefore, it is not a regular surface.[1]

5. Without loss of generality, we may assume $p(t) = (0, 0, t)$ and $q(t) = (a, t, 0)$. Then, we get the following parametrization by connecting these points.

$$\begin{aligned} X(t, s) &= p(t) + s(q(t) - p(t)) \\ &= (0, 0, t) + s(a, t, -t) \\ &= (sa, st, t - st) \end{aligned}$$

[1]In this problem we saw that the inverse image of a critical point may or may not be a regular surface.

Setting $X = (x, y, z)$, we see

$$y(x - a) + zx = st(sa - a) + (t - st)sa = 0.$$

So, it satisfies the equation. Set $f(x, y, z) = y(x - a) + zx$ then

$$f_y = x - a$$
$$f_z = x.$$

Since $a \neq 0$, the differential of f never vanishes. Therefore, all values are regular. In particular, $f^{-1}(0)$ is a regular surface by Proposition 2.