Homework 2 - Solutions

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1. (a) We need to show that $|\frac{d\alpha}{ds}| = 1$. For convenience, set $s' = \frac{s}{c}$. It is easy to see that

$$\alpha'(s) = \frac{1}{c}\left(-a\sin(s'), a\cos(s'), b\right).$$

So,

$$\begin{aligned}
\left|\alpha'(s)\right|^2 &= \left|\frac{1}{c}\left(-a\sin(s'), a\cos(s'), b\right)\right|^2 \\
&= \frac{1}{c^2}\left(a^2\sin^2(s') + a^2\cos^2(s') + b^2\right) \\
&= \frac{1}{c^2}\left(a^2\left(\sin^2(s') + \cos^2(s')\right) + b^2\right) \\
&= \frac{a^2+b^2}{c^2} \\
&= 1
\end{aligned}$$

since $c^2 = a^2 + b^2$. Therefore the curve is parametrized by arc length.

(b) Recall that the curvature is given by $k(s) = |\alpha''(s)|$.

$$\alpha''(s) = \frac{1}{c^2}\left(-a\cos(s'), -a\sin(s'), 0\right)$$

So,

$$\begin{aligned}
\left|\alpha''(s)\right|^2 &= \frac{a^2}{c^4}\left(\cos^2(s') + \sin^2(s')\right) \\
&= \frac{a^2}{c^4}
\end{aligned}$$

and $k(s) = a/c^2$.

To compute torsion we need to compute the derivative of the binormal vector $b'(s)$. Recall that $b'(s) = t(s) \times n'(s)$ where $t(s) = \alpha'(s)$ and $n(s)$ is the normal vector. Since $\alpha'' = kn$,

$$n(s) = \frac{\alpha''(s)}{k(s)} = \left(-\cos(s'), -\sin(s'), 0\right). \quad (1)$$

Therefore,

$$n'(s) = \frac{1}{c}\left(\sin(s'), -\cos(s'), 0\right).$$

Next, we take the cross product $t(s) \times n'(s)$. Note that $t(s) = -an'(s) + \frac{b}{c}e_3$. Therefore, $t(s) \times n'(s) = \frac{b}{c}e_3 \times n'(s)$. Since $e_3 \times e_1 = e_2$ and $e_3 \times e_2 = -e_1$, we get

$$b'(s) = \frac{b}{c^2}\left(\cos\left(s'\right), \sin\left(s'\right), 0\right).$$

Recall that torsion is defined by $b'(s) = \tau(s)n(s)$. So,

$$\tau(s) = -\frac{b}{c^2}.$$

(c) Note that the binormal vector $b(s)$ is a normal vector for the osculating plane at $\alpha(s)$. Therefore, its equation is given by

$$(P - \alpha(s)) \cdot b(s) = 0 \tag{2}$$

where $P(x, y, z) \in \mathbb{R}^3$. Thus, we have to compute $b(s) = t(s) \times n(s)$. By using component description of the cross product, we can easily see that

$$\begin{aligned} b(s) &= -\frac{1}{c}\left(-b\sin(s'), b\cos(s'), -a\left(\sin^2(s') + \cos^2(s')\right)\right) \\ &= -\frac{1}{c}\left(-b\sin(s'), b\cos(s'), -a\right) \\ &= \frac{1}{c}\left(b\sin(s'), -b\cos(s'), a\right) \end{aligned}$$

If we plug this in (2), we get

$$\begin{aligned} 0 &= ((x, y, z) - (a\cos\left(s'\right), a\sin\left(s'\right), bs')) \cdot \left(\frac{1}{c}\left(b\sin(s'), -b\cos(s'), a\right)\right) = 0 \\ &= \left(x - a\cos\left(s'\right), y - a\sin\left(s'\right), z - bs'\right) \cdot \left(b\sin(s'), -b\cos(s'), a\right) \\ &= xb\sin(s') - yb\cos(s') + az - ab\cos(s')\sin(s') + ab\cos(s')\sin(s') - abs' \\ &= xb\sin(s') - yb\cos(s') + az - abs' \end{aligned}$$

or

$$abs' = xb\sin(s') - yb\cos(s') + az.$$

(d) Let L be the line passing through $\alpha(s)$ with the direction vector $n(s)$. As the direction vector for the z-axis, we may take e_3. From (1), it is clear that $n(s) \cdot e_3 = 0$. In other words, if these two lines intersect, they meet at a constant angle of $\pi/2$. So, all we have to show is that these two lines indeed intersect.

A parametric equation of a line passing through $p_0 \in \mathbb{R}^3$ with direction vector v is given by $t \mapsto p_0 + tv$. Taking $p_0 = \alpha(s)$ and $v = n(s)$, we get a parametric equation of L

$$L(t) = (a\cos(s'), a\sin(s'), bs') + t\left(-\cos(s'), -\sin(s'), 0\right).$$

Now, it is clear that $L(a) = (0, 0, bs')$ which is a point on the z-axis.

(e) Here, we only need to compare the direction vectors, $\alpha'(s)$ and e_3. Taking the dot product of the two we get $\alpha'(s) \cdot e_3 = \frac{b}{c}$ which is constant. So, therefore, the angle between them is constant.

2. First, we need to check if α is parametrized by arc length! An immediate application of the fundamental theorem of calculus (FTC) gives us

$$\alpha'(s) = (\cos(\theta(s)), \sin(\theta(s)))$$

which is clearly a unit vector. Therefore, $\alpha(s)$ is parametrized by arc length. Hence, the curvature is given by $k(s) = |\alpha''(s)|$. So, we compute

$$\begin{aligned}\alpha''(s) &= (-\sin(\theta(s))\theta'(s), \cos(\theta(s))\theta'(s)) \\ &= \theta'(s)\left(-\sin(\theta(s)), \cos(\theta(s))\right) \\ &= k(s)\left(-\sin(\theta(s)), \cos(\theta(s))\right)\end{aligned}$$

where we used FTC again to figure out $\theta'(s)$. Now, it is clear that $\alpha''(s)$ is of the form $k(s)$ times a unit vector. Therefore, the curvature of $\alpha(s)$ is $k(s)$[1].

Say $\beta(s)$ ($s \in I$) is a plane curve parametrized by arc length and its curvature is $k(s)$. Then, we know that $\beta'(s)$ is a unit vector for all $s \in I$. Thus, it can be expressed as $\beta'(s) = (\cos(\tau(s)), \sin(\tau(s)))$ for some function $\tau(s)$[2]. Then,

$$\beta''(s) = \tau'(s)\left(-\sin(\tau(s)), \cos(\tau(s))\right).$$

Since the curvature of $\beta(s)$ is $k(s)$, we have $\tau'(s) = k(s)$. Therefore, $\tau(s) = \int k(s)ds + C$. Moreover,

$$\beta(s) = \int \beta'(s)ds = \left(\int \cos(\tau(s))ds + A, \int \sin(\tau(s))ds + B\right).$$

3. (a) Using the product rule we get

$$\begin{aligned}\alpha'(t) &= e^t\left(\cos(t), \sin(t)\right) + e^t(-\sin(t), \cos(t)) \\ &= e^t\left(\cos(t) - \sin(t), \cos(t) + \sin(t)\right).\end{aligned}$$

Thus,

$$\begin{aligned}|\alpha'(t)|^2 &= e^{2t}\left(\cos^2(t) - 2\cos(t)\sin(t) + \sin^2(t) + \cos^2(t) + 2\cos(t)\sin(t) + \sin^2(t)\right) \\ &= e^{2t}2\left(\cos^2(t) + \sin^2(t)\right) \\ &= 2e^{2t}\end{aligned}$$

and $|\alpha'(t)| = \sqrt{2}e^t$. Clearly, this function is always positive but not identically zero. Thus, α is regular but not parametrized by arc length.

(b) By the FTC, $s' = f'(t) = |\alpha'(t)|$.

(c) Since $s' = |\alpha'(t)| > 0$, s is an increasing function.

[1] Actually, in this problem, we can talk about signed curvature and everything still holds. We just have to note that $\alpha'(s)$ and $\alpha''(s)$ form a positively oriented basis.

[2] Here, we need to prove that τ can be choosen as a differentiable function. It is in fact possible because β is differentiable and on small intervals we can apply $\cos^{-1}$ or $\sin^{-1}$ as necessary but we will skip this part of the proof.

(d)

$$\begin{aligned} f(t) &= \int_0^t |\alpha'(u)| du \\ &= \int_0^t \sqrt{2} e^u du \\ &= \sqrt{2} e^u \Big|_0^t \\ &= \sqrt{2}(e^t - 1) \end{aligned}$$

(e) The domain of $f(t)$ is all of $\mathbb{R}$ and the range is $(-\sqrt{2}, \infty)$.

(f)

$$\begin{aligned} s &= \sqrt{2}(e^t - 1) \\ \frac{s}{\sqrt{2}} &= e^t - 1 \\ e^t &= \frac{s}{\sqrt{2}} + 1 \\ t &= \ln\left(\frac{s}{\sqrt{2}} + 1\right) \\ t &= f^{-1}(s) = \ln\left(\frac{s}{\sqrt{2}} + 1\right) \end{aligned}$$

The domain of f^{-1} is $(-\sqrt{2}, \infty)$ and the range is $\mathbb{R}$.

(g)

$$\begin{aligned} \text{trace}(\beta) &= \left\{\beta(s) \mid s \in (-\sqrt{2}, \infty)\right\} \\ &= \left\{\alpha(f^{-1}(s)) \mid s \in (-\sqrt{2}, \infty)\right\} \\ &= \{\alpha(t) \mid t \in \mathbb{R}\} \\ &= \text{trace}(\alpha). \end{aligned}$$

(h) By the chain rule and the derivative of the inverse function,

$$\begin{aligned} \beta'(s) &= \alpha'(f^{-1}(s)) \left(f^{-1}(s)\right)' \\ &= \alpha'(f^{-1}(s)) \left(\frac{1}{f'\left(f^{-1}(s)\right)}\right) \\ &= \alpha'(t) \left(\frac{1}{f'(t)}\right) \\ &= \alpha'(t) \left(\frac{1}{|\alpha'(t)|}\right) \end{aligned}$$

by part (b). So, we already see that

$$|\beta'(s)|^2 = \left|\frac{\alpha'(t)}{|\alpha'(t)|}\right|^2 = 1.$$

Continuing with the previous calculation we get

$$\begin{aligned}
\beta'(s) &= \frac{\alpha'(t)}{|\alpha'(t)|} \\
&= \frac{e^t}{\sqrt{2}e^t}\left(\cos(t) - \sin(t), \cos(t) + \sin(t)\right) \\
&= \frac{1}{\sqrt{2}}\left(\cos(t) - \sin(t), \cos(t) + \sin(t)\right) \\
&= \frac{1}{\sqrt{2}}\left(\cos(f^{-1}(s)) - \sin(f^{-1}(s)), \cos(f^{-1}(s)) + \sin(f^{-1}(s))\right) \\
&= \frac{1}{\sqrt{2}}\left(\cos(\ln\left(\frac{s}{\sqrt{2}}+1\right)) - \sin(\ln\left(\frac{s}{\sqrt{2}}+1\right)), \cos(\ln\left(\frac{s}{\sqrt{2}}+1\right)) + \sin(\ln\left(\frac{s}{\sqrt{2}}+1\right))\right)
\end{aligned}$$

(i) Yes, since $\beta'(s)$ is a unit vector by the previous part.

4. We present two solutions assuming α is parametrized by arc length.

(a) Solution 1: Define $d(t) = \frac{1}{2}|\alpha(t)|^2 = \frac{1}{2}\alpha(t)\cdot\alpha(t)$. If the distance is maximized at $t = 0$, d is also maximized at $t = 0$. Therefore, by the second derivative test, we have $d''(0) \leq 0$. Note that $d'(t) = \alpha'(t)\cdot\alpha(t)$. Thus,

$$d''(t) = \alpha''(t)\cdot\alpha(t) + \alpha'(t)\cdot\alpha'(t)$$

Since α is parametrized by arc length, $\alpha'(t)\cdot\alpha'(t) = 1$. Hence,

$$d''(t) = \alpha''(t)\cdot\alpha(t) + 1.$$

By the second derivative test at $t = 0$, we get

$$0 \geq d''(0) = \alpha''(0)\cdot\alpha(0) + 1$$

or

$$\alpha''(0)\cdot\alpha(0) \leq -1.$$

Since $\alpha''(0) = k(0)n(0)$, this is equivalent to

$$\begin{aligned}
-1 \geq& k(0)n(0)\cdot\alpha(0) \\
&= k(0)|n(0)||\alpha(0)|\cos(\theta) \\
&= k(0)|\alpha(0)|\cos(\theta).
\end{aligned}$$

where we used that n is unit. Now, all the terms in the last line are positive except for $\cos(\theta)$. So, we must have $\cos(\theta) < 0$. Therefore,

$$k(0) \geq \frac{-1}{|\alpha(0)|\cos(\theta)}.$$

However, since $-1 \leq \cos(0) < 0$ implies $\frac{-1}{\cos(\theta)} \geq 1$, we also have

$$k(0) \geq \frac{-1}{|\alpha(0)|\cos(\theta)} \geq \frac{1}{|\alpha(0)|}.$$

(b) Solution 2 using coordinates: (Assuming α is parametrized by arc length.) We identify $(x, y) \in \mathbb{R}^2$ with $x + iy \in \mathbb{C}$ in the usual way. Every complex number can be written as $re^{i\theta}$ for some $r \geq 0$ and $\theta \in \mathbb{R}$ by Euler's identity. So, in particular, we may assume that the curve α is given by $\alpha(t) = r(t)e^{i\theta(t)}$ for some real differentiable functions r and θ. Since α is regular, we may reparametrize it by arc length. We assume that we have reparametrized it but we will continue to use t as our parameter. Note that we may further assume that $|\alpha(t)| = r(t)$ is maximized at $t_0 = 0$ and

$$\theta(0) = 0$$

(by rotating the plane if necessary) without loss of generality. Let

$$R = r(0).$$

Since r has a local maximum at 0, we have

$$r'(0) = 0$$

and

$$r''(0) \leq 0.$$

The curvature of α at $t = 0$ is $|\alpha''(0)|$. So, we want to show that $|\alpha''(0)| \geq \frac{1}{R}$. In the following calculations we will surpress t from our notation to avoid cluttering. Since $\alpha = re^{i\theta}$, we get

$$\alpha' = r'e^{i\theta} + ir\theta' e^{i\theta}.$$

Note that to compute (the square of) the norm of a complex number $x + iy$, we multiply it by its conjugate;

$$|x + iy|^2 = \left(\overline{x + iy}\right)(x + iy) = (x - iy)(x + iy) = x^2 + y^2.$$

Since $|\alpha'|^2 = 1$, we have

$$\begin{aligned} 1 &= \overline{\alpha'}\alpha \\ &= \left(r'e^{-i\theta} - ir\theta' e^{-i\theta}\right)\left(r'e^{i\theta} + ir\theta' e^{i\theta}\right). \end{aligned}$$

Here, we plug in $t = 0$ to simplify this and get

$$1 = (R\theta'(0))^2.$$

So, we see

$$\theta'(0) = \pm\frac{1}{R}.$$

Now, we continue to take derivative once more.

$$\begin{aligned} \alpha'' &= r''e^{i\theta} + 2ir'\theta' e^{i\theta} + ir(\theta' e^{i\theta})' \\ &= r''e^{i\theta} + 2ir'\theta' e^{i\theta} + ir(\theta'' e^{i\theta} + i\theta'^2 e^{i\theta}) \end{aligned}$$

We plug in $t = 0$ to simplify the expression first;

$$\begin{aligned}\alpha''(0) &= r''(0) + iR(\theta''(0) + i\frac{1}{R^2}) \\ &= r''(0) - \frac{1}{R} + iR\theta''(0).\end{aligned}$$

So,

$$\begin{aligned}k(s) &= |\alpha''(0)| \\ &\geq |\text{real part}(\alpha''(0))| \\ &= \left|r''(0) - \frac{1}{R}\right| \\ &= -r''(0) + \frac{1}{R} \\ &\geq \frac{1}{R}\end{aligned}$$

as $r''(0) \leq 0$.

5. Since the cross product $u \times v$ of two vectors u and v is always perpendicular to u and v, the "only if" part is clear.

So, we proceed to show that if $v \neq 0$ and w are perpendicular, there exists a vector u such that $u \times v = w$. Let $c = 1/|v|^2$ and $u = cv \times w$ (note that c is well-defined since $v \neq 0$). Then, by problem 4 of the first homework set, we have

$$\begin{aligned}u \times v &= c(v \times w) \times v \\ &= c\left[(v \cdot v)w - (w \cdot v)v\right] \\ &= c|v|^2 w = w\end{aligned}$$

since w and v are perpendicular ($w \cdot v = 0$).

Let u' be another vector such that $u' \times v = w$. Then, by linearity we have

$$\begin{aligned}(u' - u) \times v &= u' \times v - u \times v \\ &= w - w = 0.\end{aligned}$$

However, we know that the cross product of two vectors is 0 if and only if they are linearly dependent. Since, $v \neq 0$, we must have $u' - u = kv$ for some $k \in \mathbb{R}$. Therefore, the most general solution is $u' = kv + \frac{1}{|v|^2} v \times w$.