Exercise set 2

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These are just some exercise problems they are not going to be graded.

1. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $F(x, y) = (x + 2y, e^y + 3x)$. Define the following curves

$$\alpha(t) = (2t, 0) \text{ for } t \in \mathbb{R}$$
$$\beta(t) = (t^2 + 2t, \cos(t) - 1) \text{ for } t \in \mathbb{R}$$
$$\gamma(t) = (2\sin(t), \ln(t + 1) - \sin(t)) \text{ for } t \in (-1, \infty).$$

   (a) Compute $\alpha(0), \beta(0), \gamma(0)$.
   (b) Compute $\alpha'(0), \beta'(0), \gamma'(0)$.
   (c) Compute the tangent vector of the curves $F(\alpha(t)), F(\beta(t))$, and $F(\gamma(t))$ at $t = 0$.
   (d) Compute the partial derivative of F with respect to x at $(x, y) = (0, 0)$.
   (e) Do you see any relation between your answers to the last two parts?

2. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $F(x, y) = (x + 2y, e^y + 3x)$.

   (a) Compute its Jacobian matrix.
   (b) Where (in the xy-plane) is the Jacobian matrix non-invertible?

3. Define $F : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$F(r, \theta, \varphi) = (r\sin(\theta)\cos(\varphi), r\sin(\theta)\sin(\varphi), r\cos(\theta)).$$

   (a) Show that $|F(r, \theta, \varphi)| = |r|$. Note that in particular, the function $f : \mathbb{R}^2 \to \mathbb{R}^3$ defined as the restriction $f(\theta, \varphi) = F(1, \theta, \varphi)$ has image contained in the unit sphere $S \subset \mathbb{R}^3$ where

$$S = \left\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\right\}.$$

   (b) What is the image of $F(r, 0, \varphi)$?
   (c) What is the image of $F(r, \pi/2, \varphi)$?
   (d) What is the image of $F(r, \pi/3, \varphi)$?
   (e) What is the image of $F(r, \theta, 0)$?
   (f) What is the image of $F(r, \theta, \pi/2)$?
   (g) What is the image of $F(r, \theta, \pi/3)$?