Associative and Cayley Grassmannians

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Overview

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    - Exceptional cross products
2. Background
    - Associative and Cayley Grassmannians
    - Plücker embedding and charts of $Gr(4, 8)$
3. Results
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    - Singular locus

Riemannian manifolds with exceptional holonomy

In 1955, M. Berger came up with a list of possible Riemannian holonomy groups. The last open cases were the *exceptional* groups G_2 and $Spin(7)$.

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There exists (non-compact) complete Riemannian manifolds with exceptional holonomy.

Theorem (D. Joyce '96)

There exists compact (and complete) Riemannian manifolds with exceptional holonomy.

Riemannian manifolds with exceptional holonomy

These holonomy groups allows us to define calibration forms.

In dimension 7, we have a global 3-form φ called the associative calibration and 3 dimensional calibrated submanifolds are called the associative submanifolds.

In dimension 8, we have a global 4-form Φ called the Cayley calibration and 4 dimensional calibrated submanifolds are called Cayley submanifolds.

Calibrated submanifolds are absolutely volume minimizing submanifolds in their homology classes.

Exceptional cross products

Definition

Let (V, B) be an inner product space of dimension n (over $\mathbb{R}$ or $\mathbb{C}$). A multilinear map $L : V^r \to V$ is called an r-fold cross product if

1. $B(L(v_1, \cdots, v_r), v_i) = 0 \qquad$ for $i = 1, \cdots, r$
2. $N(L(v_1, \cdots, v_r)) = N(v_1 \wedge \cdots \wedge v_r)$

where N is the induced quadratic form i.e. $N(v) = B(v, v)$.

Theorem (R. Brown and A. Gray '67)

A r-fold cross product exists only in the following cases

1. $r = 1$ *and* n *is even*
2. $r = n - 1$ *and* n *is arbitrary*

   The exceptional cases:
3. $r = 2$ *and* $n = 7$
4. $r = 3$ *and* $n = 8$

Associative Grassmannian

Definition

A 3-plane in V (of dimension 7) is called associative if it is closed under the (2-fold) cross product. The set of all associative planes is called the associative Grassmannian.

G_2 acts transitively on the associative Grassmannian with stabilizer group $SO(4)$.

More on the associative Grassmannian can be found in [Harvey, Lawson '82], [Shi, Zhou '14], [Akbulut, Kalafat '15] and [Akbulut, Can '15].

Cayley Grassmannian

Definition

A 4-plane in V (of dimension 8) is called Cayley if it is closed under the (3-fold) cross product. The set of all Cayley planes is called the Cayley Grassmannian.

$Spin(7)$ acts transitively on the Cayley Grassmannian with stabilizer group $K = (SU(2) \times SU(2) \times SU(2)) / \pm 1$.

In fact, over $\mathbb{R}$, $Spin(7)/K = SO(7)/(SO(3) \times SO(4)) = Gr(3,7)$.

For the rest of the talk, we will consider the Cayley Grassmannian over $\mathbb{C}$.

Octonions and cross products

A normed algebra (A, N) is called a composition algebra if

$$N(uv) = N(u)N(v).$$

Fact

There exists a unique 8 dimensional composition algebra $\mathbb{O}$ over $\mathbb{C}$ called the octonion algebra.

We call the span of 1 the real part and the orthogonal complement $Im(\mathbb{O}) = \langle 1 \rangle^{\perp}$ the imaginary part.

Definition (Cross products)

For $u, v, w \in Im(\mathbb{O})$,

$$u \times v = Im(uv).$$

For u, v, w, z,

$$u \times v \times w = \frac{1}{2}\left((u\bar{v})w - (w\bar{v})u\right).$$

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This allows us to define the calibration forms

$$\varphi(u, v, w) = B(u, v \times w)$$

$$\Phi(z, u, v, w) = B(z, u \times v \times w)$$

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Fact (Cayley Equality)

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A 4-plane ξ generated by orthonormal z, u, v, w is called Cayley if $\Phi(z, u, v, w) = \pm 1$ (or equivalently $\Xi(z, u, v, w) = 0$).

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Note that $\Xi \in \Lambda^4 \mathbb{O}^* \otimes Im(\mathbb{O})$ i.e. it gives us 7 linear equations on $\Lambda^4 \mathbb{O}$.

Plücker embedding

$$Gr(4,8) \hookrightarrow \mathbb{P}(\Lambda^4\mathbb{C}^8)$$

$$\xi = \langle z, u, v, w\rangle \mapsto [z \wedge u \wedge v \wedge w]$$

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Minimal compactification of the Cayley Grassmannian:

$$X_{min} = Gr(4,8) \cap \{\Xi = 0\}$$

Charts of Gr(4,8)

Let $[p_{0123} : p_{0124} : \cdots : p_{4567}]$ denote the homogenous coordinates on $\mathbb{P}(\Lambda^4(\mathbb{C}^8))$. On the chart $U_{stun} = \{p_{stun} \neq 0\}$, coordinate functions (of $Gr(4,8)$) are

$$q_{ijkl} = \frac{p_{ijkl}}{p_{stun}}$$

for $|\{i,j,k,l\} \cap \{s,t,u,n\}| = 3$ i.e. index sets differ by exactly one element.

Torus action on X_{min}

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For $\lambda \in \mathbb{C}^*$, let

$$L_\lambda = \begin{pmatrix} P_\lambda & -iM_\lambda \\ iM_\lambda & P_\lambda \end{pmatrix} \in SO(2, \mathbb{C})$$

where $P_\lambda = \frac{\lambda+\lambda^{-1}}{2}$ and $M_\lambda = \frac{\lambda-\lambda^{-1}}{2}$. Then,

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$h(\lambda,\mu,\gamma) = diag(L_{\lambda\mu}, L_{\lambda\mu^{-1}}, L_{\lambda\gamma}, L_{\lambda\gamma^{-1}}) \in Spin(7,\mathbb{C})$.

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(Maximal) Torus $T = Image(h) \cong (\mathbb{C}^*)^3$.

Three $SL(2,\mathbb{C})$ actions

For $g \in SL(2,\mathbb{C}) \cong \{q \in \mathbb{H} \mid N(q) = 1\} = S^3_{\mathbb{C}}$, and $(x, y) \in \mathbb{O} = \mathbb{H} \oplus \mathbb{H}$,

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These actions define three embeddings of $SL(2,\mathbb{C})$ into $Spin(7,\mathbb{C})$. In fact, they all map into the stabilizer of $\langle e_0, e_1, e_2, e_3 \rangle$.

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Proposition (Y.)

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Proof.

Let W be a weightspace of dimension greater than 1. Let $x = \sum_I c_I p_I$ be a non-trivial linear combination of weigthvectors. Evaluate x on a Plücker relation to show that $[x] \notin Gr(4,8)$. □

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Yes.

Singular locus Σ

Theorem (Y.)

Σ is (rational) cohomology $\mathbb{P}^5$.

Proof.

Σ is Kähler and of dimension 5. Thus, its even betti numbers are at least 1. Bialynicki-Birula decomposition implies that the sum of all betti numbers can be at most $\#\Sigma^T = 6$. □

Singular locus Σ

In fact, $\Sigma = G_2/P_\beta$.

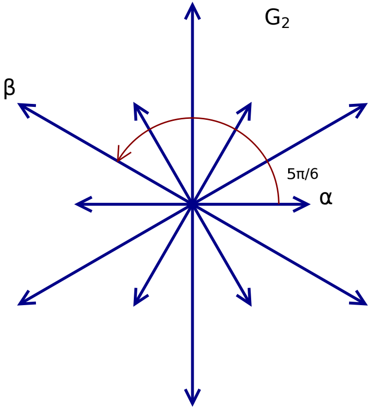

Figure: Root diagram of G_2

Source: https://commons.wikimedia.org/wiki/File:Root_system_G2.svg

References

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On the minimal compactification of the Cayley Grassmannian
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Thank you