G_2-Manifolds

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Abstract

In this mini course series, we will cover the very basics of G_2-geometry with occasional mentions of analogous statements for Spin(7). We will start with a definition of the group G_2, talk about some of its representation theoretic aspects. Then, more geometric aspects will be discussed: a G_2-manifold is a 7-manifold with a structure group isomorphic to G_2 that satisfies certain geometric conditions. These geometric conditions allow one to investigate the volume minimizing submanifolds in different ways. Towards the end of the mini course series, we will also discuss their complexifications and connections with symplectic manifolds.

1 The Group

1.1 Octonions

Definition 1.1. Octonions $\mathbb{O}$ are an 8 dimensional unital algebra with a (non-degenerate) symmetric billinear form B that satisfies

$$N(uv) = N(u)N(v) \tag{1}$$

for all $u, v \in \mathbb{O}$ where $N(u) = B(u, u)$.

Remark 1.2. Octonions are necessarily non-associative but they are alternative (i.e. any subalgebra generated by two elements is associative). See [SV13]

Remark 1.3. Over $\mathbb{R}$ there are only two such algebras (up to isomorphism) and over $\mathbb{C}$ there is a unique such algebra.

We will usually work over $\mathbb{R}$ with a positive definite B but octonions over $\mathbb{C}$ will also be an important for our purposes at certain times.

Warning 1.4. We will sometimes call B a metric and N norm even though B is not a metric over $\mathbb{C}$ and N is actually a quadratic form (it is more like norm-squared).

Lemma 1.5. For $u, v, v' \in \mathbb{O}$,

$$N(u)B(v, v') = B(uv, uv') = B(vu, v'u). \tag{2}$$

In particular, for unit u, (left or right) multiplication by u is an orthogonal transformation of $\mathbb{O}$.

Proof. Since $B(v, v') = \frac{1}{2}(N(v + v') - N(v) - N(v'))$, we have

$$\begin{aligned} N(u)B(v, v') &= \frac{1}{2}\big(N(u)N(v + v') - N(u)N(v) - N(u)N(v')\big) \\ &= \frac{1}{2}\big(N(uv + uv') - N(uv) - N(uv')\big) \\ &= B(uv, uv'). \end{aligned}$$

The second equality can be proved similarly. □

Example 1.6. Consider the (real or complex) vector space $\mathbb{O}$ generated by $S = \{1, \mathbf{i}, \mathbf{j}, \mathbf{k}, \mathbf{l}, \mathbf{li}, \mathbf{lj}, \mathbf{lk}\}$. Define B so that S is orthonormal and define multiplication as follows.

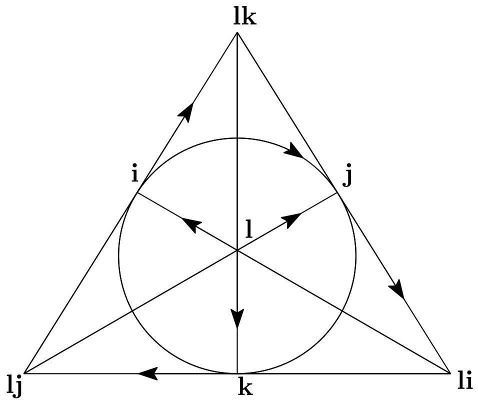

Figure 1: Multiplication table for octonions

For each (oriented) line (or the circle) in Figure 1 from x to y to z, set

$$xy = z = -yx, \qquad yz = x = -zy, \qquad zx = y = -xz,$$

and

$$x^2 = y^2 = z^2 = -1.$$

Over $\mathbb{C}$, this gives us the unique complex octonion algebra and over $\mathbb{R}$ it gives us the unique octonion algebra with positive definite inner product.

Definition 1.7. We define the real part $\mathrm{Re}(\mathbb{O})$ of $\mathbb{O}$ to be the span of 1 and the imaginary part $\mathrm{Im}(\mathbb{O})$ to be the orthogonal complement of 1. Of course, this allows us to decompose a given element of $\mathbb{O}$ into two parts and we use the same notation for this decomposition. This decomposition allows us to define conjugation by

$$\overline{u} = \mathrm{Re}(u) - \mathrm{Im}(u). \tag{3}$$

Exercise 1.8.

$$\overline{uv} = \overline{v}\,\overline{u} \tag{4}$$

and therefore, $\overline{\overline{u}u} = \overline{u}u \in \mathrm{Re}(\mathbb{O})$.

Exercise 1.9.

$$B(u, v) = \mathrm{Re}(\overline{u}v) \tag{5}$$

Definition 1.10. The group G_2 is defined to be the automorphism group of $\mathbb{O}$.

Clearly, G_2 fixes the real part of octonions.

Proposition 1.11. $\mathrm{G}_2 \leq O(\mathbb{O}, B)$ (the orthogonal group of $\mathbb{O}$).

Proof. Given $A \in \mathrm{G}_2$,

$$\begin{aligned} B(Au, Av) &= \mathrm{Re}(\overline{Au}Av) \\ &= \mathrm{Re}(A\overline{u}Av) \\ &= \mathrm{Re}(A(\overline{u}v)) \\ &= \mathrm{Re}(\overline{u}v) \\ &= B(u, v) \end{aligned}$$

□

Remark 1.12. Since G_2 fixes $\mathrm{Re}(\mathbb{O})$, we can also think of G_2 as a subgroup of $O(\mathrm{Im}(\mathbb{O}), B)$.

Definition 1.13. $(u, v, w) \in (\mathrm{Im}(\mathbb{O}))^3$ is called a G_2-triple if $\{u, v, uv, w\}$ is orthonormal.

Theorem 1.14. G_2 is homeomorphic to the set of all G_2-triples.

Proof. We define maps from G_2 to G_2-triples and back.

Since $A \in \mathrm{G}_2$ is an orthogonal transformation and $\{\mathbf{i}, \mathbf{j}, \mathbf{k}, \mathbf{l}\}$ is orthonormal, $\{A\mathbf{i}, A\mathbf{j}, A\mathbf{k}, A\mathbf{l}\}$ is orthonormal and therefore, $(A\mathbf{i}, A\mathbf{j}, A\mathbf{k})$ is a G_2-triple.

On the other hand, if we have a G_2-triple (u, v, w), there is a unique algebra homomorphism that sends $\mathbf{i}$ to u, $\mathbf{j}$ to v and $\mathbf{l}$ to w. It is easy to see that this unique map is in fact an isomorphism. Hence, it defines an element of G_2.

Clearly, these two correspondences are inverse to each other and they are continuous. □

Proposition 1.15. If $u, v \in \mathrm{Im}(\mathbb{O})$ and perpendicular, then $uv \in \mathrm{Im}(\mathbb{O})$. Moreover, uv is perpendicular to both u and v.

Proof.

$$\begin{aligned} \mathrm{Re}(uv) &= B(\overline{u}, v) \\ &= -B(u, v) \\ &= 0 \end{aligned}$$

proves the first part.

The second part, follows immediately from Lemma 1.5. □

Corollary 1.16. The set of all G_2-triples (and therefore, the group G_2 itself) is an S^3 fibration over $V_{2,7}$ the Stiefel manifold of orthonormal 2-frames in 7-space.

Proof. Recall that (u, v, w) is a G_2-triple if $\{u, v, uv, w\}$ is orthonormal. So, we are free to choose any unit imaginary octonion as u. Then, v has to be a unit imaginary octonion that is perpendicular to u. So the map $(u, v, w) \mapsto (u, v)$ is in fact a projection onto the Stiefel manifold $V_{2,7}$.

Then, by Proposition 1.15, uv is also a unit imaginary octonion. So, the fiber of the projection is S^3. □

Warning 1.17. Note that analogous statement is true for complex G_2 (which is the automorphism group of complex octonions). However, in that case one has to describe orthonormality with respect to B. In particular, the base space is not the Stiefel manifold but its "complexification".

Corollary 1.18.
1. $\mathrm{G}_2 \leq SO(\mathrm{Im}(\mathbb{O})) \cong SO(7)$
2. $\dim(\mathrm{G}_2) = 14$
3. G_2 is 2-connected.
4. G_2 is compact.

Proof. $V_{2,7}$ is an S^5 fibration over S^6. So in particular it is 11 dimensional and 4-connected. Therefore, $\mathrm{G}_2 \leq O(\mathrm{Im}(\mathbb{O}))$ being an S^3 fibration over $V_{2,7}$ satisfies all the listed properties. □

Corollary 1.19. G_2 acts transitively on $V_{2,7}$.

1.2 Cross product operations

Definition 1.20. A multi linear map $L : V^r \to V$ on an inner product space is called an r-fold cross product if the output is perpendicular to all the input vectors and the norm of the output is equal to the norm of the wedge product of the inputs. More precisely, the following two conditions are required:

1. $B(v_i, L(v_1, \cdots, v_r)) = 0$ for all $1 \le i \le r$ and
2. $N(L(v_1, \cdots, v_r)) = N(v_1 \wedge \cdots \wedge v_r)$ where the norm on $\Lambda^r V$ is the induced norm.

Exercise 1.21. If L is an alternating map, then the second condition is equivalent to

$$N(L(v_1, \cdots, v_r)) = N(v_1) \cdots N(v_r)$$

for all orthogonal vectors $v_1, \cdots, v_r$.

Theorem 1.22 ([BG67])**.** r-fold cross product on an n-dimensional inner product space exists only in the following cases:

1. n is even, $r = 1$
2. n is arbitrary, $r = n - 1$
3. $n = 7$ and $r = 2$
4. $n = 8$ and $r = 3$

Remark 1.23. The last two cases are specific to dimensions 7 and 8 only. Because of this, these are sometimes called exceptional cases.

Definition 1.24. We define a (2-fold) cross product operation on $\mathrm{Im}(\mathbb{O})$ as follows:

$$u \times v = \mathrm{Im}(uv)$$

Proposition 1.25. The operation $(u, v) \mapsto u \times v$ is indeed a cross product operation in the sense of Definition 1.20.

Proof. Note that for $u \in \mathrm{Im}(\mathbb{O})$, $\overline{u} = -u$. Hence,

$$\begin{aligned} u \times u &= \mathrm{Im}(uu) \\ &= -\mathrm{Im}(\overline{u}u) \\ &= 0 \end{aligned}$$

by Exercise 1.8. This shows the operation is alternating i.e. $u \times v = -v \times u$. Thus, it is enough to prove $B(u, u \times v) = 0$. However, now this follows easily from being alternating and Proposition 1.15.

Next, Proposition 1.15 gives us

$$\begin{aligned} N(u \times v) &= N(\mathrm{Im}(uv)) \\ &= N(uv) \\ &= N(u)N(v) \end{aligned}$$

for orthogonal u and v. So, by Exercise 1.21, we are done. $\square$

Definition 1.26. Set $\varphi \in \Lambda^3(\mathrm{Im}(\mathbb{O})^*)$ to be

$$\varphi(u, v, w) = B(u, v \times w).$$

We call φ the associative calibration.

Remark 1.27. Note that Proposition 1.25 shows that φ indeed is an alternating 3-form on $\mathrm{Im}(\mathbb{O})$.

Proposition 1.28. Given $A \in \mathrm{G}_2$, $A(u \times v) = Au \times Av$

Proof.

$$\begin{aligned} A(u \times v) &= A(\mathrm{Im}(uv)) \\ &= \mathrm{Im}(A(uv)) \\ &= \mathrm{Im}(AuAv) \\ &= Au \times Av \end{aligned}$$

□

Definition 1.29. Let G^{φ} be the stabilizer group of φ in $GL(\mathrm{Im}(\mathbb{O})) \cong GL(7, \mathbb{R})$.

Proposition 1.30.

$$\mathrm{G}_2 \leq G^{\varphi}$$

Proof. Given $A \in \mathrm{G}_2$, since $\mathrm{G}_2 \leq SO(\mathrm{Im}(\mathbb{O}))$,

$$\begin{aligned} (A^{-1^*}\varphi)(u, v, w) &= \varphi(Au, Av, Aw) \\ &= B(Au, Av \times Aw) \\ &= B(Au, A(v \times w)) \\ &= B(u, v \times w) \\ &= \varphi(u, v, w) \end{aligned}$$

by Proposition 1.28. □

Proposition 1.31.

$$G^{\varphi} \leq \mathrm{G}_2$$

Proof. Let $e_0 = 1, e_1 = \mathbf{i}, e_2 = \mathbf{j}, e_3 = \mathbf{k}, e_4 = \mathbf{l}, e_5 = \mathbf{li}, e_6 = \mathbf{lj}$, and $e_7 = \mathbf{lk}$ be the standard orthonormal basis of $\mathbb{O}$. Note that

$$(\iota(u)\varphi) \wedge (\iota(v)\varphi) \wedge \varphi = 6B(u, v)\mathrm{Vol} \tag{6}$$

holds for all $u, v \in \mathrm{Im}(\mathbb{O})$ where $\iota(u)\varphi$ is the contraction of φ with u, $\mathrm{Vol} = e^{12\dots7} = e^1 \wedge e^2 \wedge \dots \wedge e^7$ and (e^i) is the basis of $\mathbb{O}^*$ dual to (e_i). Thus, for $A \in G^{\varphi}$, we have

$$\begin{aligned} A^*\left((\iota(u)\varphi) \wedge (\iota(v)\varphi) \wedge \varphi\right) &= A^*\left(6B(u, v)\mathrm{Vol}\right) \\ \iota(Au)(A^*\varphi) \wedge \iota(Av)(A^*\varphi) \wedge (A^*\varphi) &= 6B(u, v)A^*\mathrm{Vol} \\ \iota(Au)\varphi \wedge \iota(Av)\varphi \wedge \varphi &= \det(A^{-1})6B(u, v)\mathrm{Vol} \\ 6B(Au, Av)\mathrm{Vol} &= \det(A^{-1})6B(u, v)\mathrm{Vol} \\ B(Au, Av) &= \det(A)^{-1}B(u, v) \end{aligned}$$

for all $u, v \in \mathrm{Im}(\mathbb{O})$. In particular,

$$\begin{aligned} \det(B(Ae_i, Ae_j)) &= \det\left(\det(A)^{-1}B(e_i, e_j)\right) \\ \det(A^TA) &= \det(A)^{-7} \\ \det(A)^2 &= \det(A)^{-7}. \end{aligned}$$

Therefore, $\det(A)^9 = 1$ or $\det(A) = 1$.

Since $\det(A) = 1$, $B(Au, Av) = B(u, v)$. This further gives us

$$\begin{aligned} B(u, Av \times Aw) &= \varphi(u, Av, Aw) \\ &= \varphi(AA^{-1}u, Av, Aw) \\ &= A^{-1^*}\varphi(A^{-1}u, v, w) \\ &= \varphi(A^{-1}u, v, w) \\ &= B(A^{-1}u, v \times w) \\ &= B(u, A(v \times w)) \end{aligned}$$

for all $u, v, w \in \text{Im}(\mathbb{O})$. Therefore, $A(u \times v) = Au \times Av$. Hence, for orthogonal $u, v \in \text{Im}(\mathbb{O})$,

$$\begin{aligned} A(uv) &= A\left(\text{Im}(uv)\right) \\ &= A(u \times v) \\ &= Au \times Av \\ &= \text{Im}(AuAv) \\ &= AuAv. \end{aligned}$$

Now, it is an easy exercise to prove $A(uv) = AuAv$ for arbitrary $u, v \in \mathbb{O}$. □

Corollary 1.32.

$$\text{G}_2 = G^{\varphi}$$

Definition 1.33. We define a three-fold cross product operation on $\mathbb{O}$ as follows:

$$u \times v \times w = \frac{1}{2}\left((u\overline{v})w - (w\overline{v})u\right)$$

Definition 1.34. Set $\Phi \in \Lambda^4\mathbb{O}^*$ to be

$$\Phi(z, u, v, w) = B(z, u \times v \times w).$$

We call Φ the Cayley calibration.

Definition 1.35. We define Spin(7) to be the stabilizer group of Φ in $GL(\mathbb{O}) \cong GL(8, \mathbb{R})$.

Fact 1.36. Spin(7) acts transitively on the unit octonions with stabilizer subgroup G_2. Therefore, it is a 21-dimensional, compact, 2-connected Lie group that acts transitively on $V_{3,8}$.

1.3 Non-degenerate three-forms

Definition 1.37. Given an alternating three form $\beta \in \Lambda^3 V^*$ on a 7-dimensional vector space V, we call β non degenerate if for any linearly independent $u, v \in V$ there exists $w \in V$ such that $\beta(u, v, w) \neq 0$.

Example 1.38. The associative calibration φ on $\text{Im}(\mathbb{O})$ is non degenerate.

Proposition 1.39. The orbit of φ under the action of $GL(7, \mathbb{R})$ is open.

Proof. Since the stabilizer group G_2 is 14 dimensional, the orbit $GL(7, \mathbb{R})\varphi/\text{G}_2$ is (49-14=35)-dimensional. However, $\dim(\Lambda^3\text{Im}(\mathbb{O})^*) = 35$. □

Remark 1.40. Indeed, there are only two open orbits of non degenerate three forms over $\mathbb{R}$ and only one open orbit of non degenerate three forms over $\mathbb{C}$. (Compare with Remark 1.3.)

Proposition 1.41. To a given non degenerate three form β, we may associate a symmetric billinear form and a volume form on V.

Proof. Consider the equation

$$\iota(u)\beta \wedge \iota(v)\beta \wedge \beta = 6b(u, v)\Omega$$

where $\Omega \in \Lambda^7 V^*$ and b is a billinear form. For a given non-zero volume form $\Omega \in \Lambda^7 V^*$, there is a unique corresponding b that satisfies the equation. Since $\Lambda^7 V^*$ is 1 dimensional, all volume forms are a multiple of some non-zero $\Omega \in \Lambda^7 V^*$. The symmetric billinear form we get corresponding to $c\Omega$ is $\frac{1}{c}b$. One can prove that $\frac{1}{c}b$ is non-degenerate. So, it is a metric (possibly with a signature). Therefore, in turn we can compute norm squared of $c\Omega$. Indeed, $(\frac{1}{c}b)(c\Omega, c\Omega) = c^9 b(\Omega, \Omega)$. Thus, there is a unique c so that $(\frac{1}{c}b)(c\Omega, c\Omega) = 1$. The reader can prove that the corresponding volume form $(c\Omega)$ and metric $(\frac{1}{c}b)$ are unique. □

Remark 1.42. The metric we obtain by this process can either be positive definite or of signature $(3,4)$. We will call β positive if the metric is positive definite. We will mostly be concerned with positive β in these lectures. However, it is important to note that the stabilizer group of β which gives us signature $(3,4)$ metric is called "split G_2" and there is an analogous theory for "split G_2" as well.

Remark 1.43. Given positive β, one can define a cross product structure on V by setting

$$b(u, v \times w) = \beta(u, v, w).$$

So, indeed a positive three form encodes all the structure $\mathrm{Im}(\mathbb{O})$ has.

2 Representation Theory

The main purpose of this section is to give a working knowledge of representation theory for some simple cases and a flavor of these techniques as can be used to understand certain questions involving G_2.

Remark 2.1. For compact simply connected Lie groups there is a one-to-one correspondence between Lie group representations and (complexified) Lie algebra representations.

2.1 Crash course on $\mathfrak{sl}(2,\mathbb{C})$-representation theory

In this section we will only work over $\mathbb{C}$. Main reference for this section is [Hum12].

Definition 2.2.

$$\mathrm{SL}(2,\mathbb{C}) = \left\{A \in \mathbb{C}^{2\times 2} \mid \det(A) = 1\right\}$$

and "its linearization" is

$$\mathfrak{sl}(2,\mathbb{C}) = \left\{A \in \mathbb{C}^{2\times 2} \mid \mathrm{tr}(A) = 0\right\}$$

Remark 2.3. We usually use

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \;\; \text{and } H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

as the generators of $\mathfrak{sl}(2,\mathbb{C})$. By abuse of terminology, we will call eigenvalues of H weights, eigenvectors weight vectors and eigenspaces weight spaces.

Definition 2.4. Given a Lie algebra $(\mathfrak{g}, [\cdot,\cdot])$, a linear map $\rho : \mathfrak{g} \to \mathrm{End}_{\mathbb{C}}(V)$ is called a Lie algebra representation if

$$\rho([X,Y]) = [\rho(X), \rho(Y)] := \rho(X)\rho(Y) - \rho(Y)\rho(X).$$

Remark 2.5. We usually suppress the notation and say V is a $\mathfrak{g}$-representation. Also, instead of $\rho(X)(v)$ we sometimes write Xv.

Example 2.6. Any Lie algebra $\mathfrak{g}$ is a $\mathfrak{g}$-representation, called the adjoint representation, given by $\rho(X)(Y) = [X,Y]$.

Definition 2.7. A representation is called irreducible if there is no proper non-trivial subspace $W \subset V$ such that $\rho(\mathfrak{g})(W) \subset W$.

Example 2.8. Note that

$$[H,X] = 2X, \; [H,Y] = -2Y, \;\; \text{and } [X,Y] = H.$$

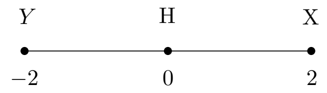

Figure 2: Weights of $\mathfrak{sl}(2,\mathbb{C})$

Example 2.9. $\mathbb{C}^2$ is an $\mathfrak{sl}(2,\mathbb{C})$-representation by usual matrix multiplication. Note that

$$He_1 = e_1, \text{ and } He_2 = -e_2.$$

Moreover,

$$\begin{aligned} Xe_2 = e_1,\ Xe_2 = 0 \\ Ye_2 = 0, \text{ and } Ye_2 = e_1. \end{aligned}$$

So, we can draw the following picture to describe the representation.

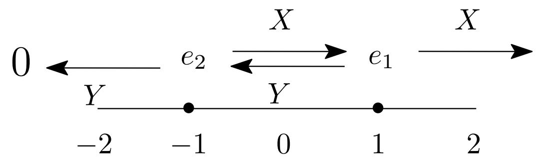

Figure 3: Weights of $\mathbb{C}^2$

Remark 2.10. Note that both of these representations are irreducible.

Proposition 2.11. Let v be a weight vector of weight n, then if Xv is non-zero, it is a weight vector of weight $n+2$. Similarly, if Yv is non-zero, it is a weight vector of weight $n-2$.

Proof.

$$HXv = [H,X]v + XHv = 2Xv + nXv = (n+2)Xv$$
$$HYv = [H,Y]v + YHv = -2Yv + nYv = (n-2)Yv$$

□

Fact 2.12 ([Hum12])**.** Let V be a finite dimensional irreducible $\mathfrak{sl}(2,\mathbb{C})$ representation. Then, there exists a weight vector v_0 of weight k such that $Xv_0 = 0$. Let $v_{i+1} = Yv_i$ for $i \geq 0$. Then, $v_i = 0$ for $i > k$, v_i is a weight vector of weight $k - 2i$ for all $0 \leq i \leq k$. $\{v_0, v_1, \cdots, v_k\}$ form a basis for V and hence, $\dim(V) = k+1$.

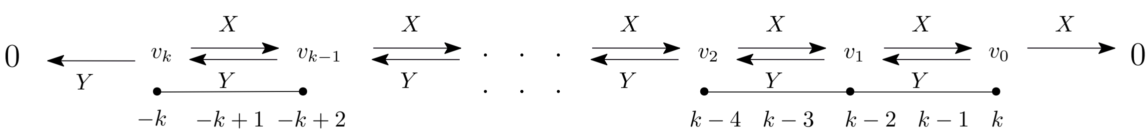

Figure 4: Finite dimensional irreducible $\mathfrak{sl}(2,\mathbb{C})$ representation of highest weight k

Fact 2.13. For every $0 \leq k \in \mathbb{Z}$, there is a unique irreducible $\mathfrak{sl}(2,\mathbb{C})$-representation of highest weight k. We denote that representation by V_k. (So, $\dim(V_k) = k+1$.)

Definition 2.14. Given two $\mathfrak{g}$-representations V and W, we can form various new representations. The dual representation V^* of V is given by

$$(Zf)(v) = -f(Zv)$$

for $Z \in \mathfrak{g}, v \in V$ and $f \in V^*$. The direct sum representation $V \oplus W$ is given by

$$Z(v \oplus w) = Zv \oplus Zw$$

for $Z \in \mathfrak{g}, v \in V$ and $w \in W$. The tensor product representation $V \otimes W$ by setting

$$Z(v \otimes w) = Zv \otimes w + v \otimes Zw$$

for $Z \in \mathfrak{g}, v \in V$ and $w \in W$.

Remark 2.15. The last definition is practically the same for alternating or symmetric tensors of V. Also, since $\mathrm{Hom}_{\mathbb{C}}(V, W) \cong V^* \otimes W$, this definition can also be used to construct $\mathrm{Hom}_{\mathbb{C}}(V, W)$ (at least in the finite dimensional case).

Notation 2.16. Let $V = \mathfrak{sl}(2, \mathbb{C})$ and $W = \mathbb{C}^2$ be the two $\mathfrak{sl}(2, \mathbb{C})$-representations defined above. We may use the following figure to describe the representation $V^2 \oplus W^3 := V \oplus V \oplus W \oplus W \oplus W$.

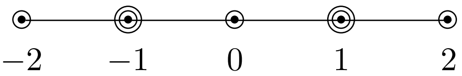

Figure 5: $V^2 \oplus W^3$

Warning 2.17. In these diagrams, a given dot (or a circle) can be used to represent a weight, the corresponding weight space or a single vector of the corresponding weight space. One should infer which interpretation is being used from the context.

Exercise 2.18. Let V and W be two $\mathfrak{sl}(2, \mathbb{C})$-representations, $v \in V$ and $w \in W$ be weight vectors of weights n and m respectively. Then, $v \otimes w$ is a weight vector of weight $n + m$.

Fact 2.19. Any finite dimensional $\mathfrak{sl}(2, \mathbb{C})$ representation is a direct sum of irreducible representations.

Example 2.20. Let $V = \mathfrak{sl}(2, \mathbb{C})$ and $W = \mathbb{C}^2$ as before. Then, the diagram for $V \otimes W$ is

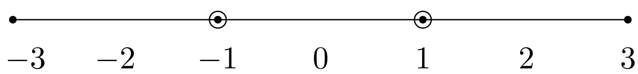

Figure 6: $V \otimes W$

Since the highest weight is 3, this representation contains a copy of V_3. If we "remove" V_3 from $V \otimes W$, we end up with a single copy of $W = V_1$. Therefore, $V \otimes W = V_1 \oplus V_3$. Another way to think about this is to note that the diagram of $V \otimes W$ is a super position of the diagrams of V_1 and V_3 (each with a single copy) and therefore, $V \otimes W = V_1 \oplus V_3$.

Example 2.21. Next, we consider $\Lambda^2 V_3$. We choose an weight vector basis of V_3 and then form distinct pairs to get a basis for $\Lambda^2 V_3$. From Exercise 2.18, we know what the weights are, for the new basis vectors and the diagram we get is the following:

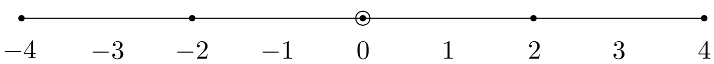

Figure 7: $\Lambda^2 V_3$

From the picture it is clear that the weigths are superposition of the weights of V_4 and V_0 with multiplicity 1. Therefore, $\Lambda^2 V_3 \cong V_0 \oplus V_4$.

2.2 $\mathfrak{g}_2$-reprensentations

This section contains a very brief discussion of some $\mathfrak{g}_2$-representation theory which avoids all the complications that arise. Here, we think of $\mathfrak{g}_2$ as the complexification of the Lie algebra of G_2. Even though, at first sight, representations of the $\mathfrak{sl}(2,\mathbb{C})$ and $\mathfrak{g}_2$ may seem different, there are great similarities. Indeed, semisimple Lie algebras contain many $\mathfrak{sl}(2,\mathbb{C})$-subalgebras. So by restriction, one can use $\mathfrak{sl}(2,\mathbb{C})$-representation theory to answer some questions about representations of other semisimple Lie algebras (including, of course, $\mathfrak{g}_2$).

For $\mathfrak{sl}(2,\mathbb{C})$, we picked specific generators and described representations according to weights with respect to H and how X and Y acts weight vectors. In the case of $\mathfrak{g}_2$, we can find two elements H_1 and H_2 in $\mathfrak{g}_2$ which commute with each other and therefore, admit simultaneous weight vectors. (Two is the maximal number for $\mathfrak{g}_2$.) This gives us a pair of weights for every common weight vector of H_1 and H_2. So, instead of using a single line to describe representations, we use a plane. Indeed, the natural action of $\mathfrak{g}_2$ on (complexified) $\mathrm{Im}(\mathbb{O})$ can be represented using the following diagram:

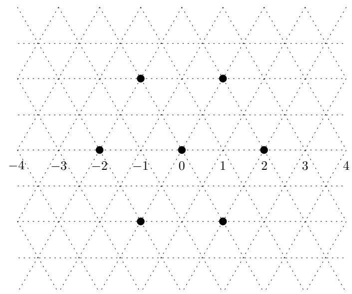

Figure 8: $\mathrm{Im}(\mathbb{O})$

The diagram of the adjoint representation is the following:

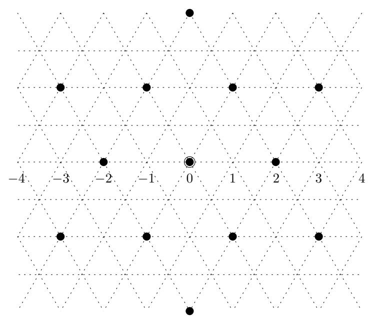

Figure 9: The adjoint representation of $\mathfrak{g}_2$

Remark 2.22. Note that we can extend Exercise 2.18 easily to the case of $\mathfrak{g}_2$.

Example 2.23. Consider the induced representation $\Lambda^2\mathrm{Im}(\mathbb{O})$. We can easily find the weight vectors and their multiplicities.

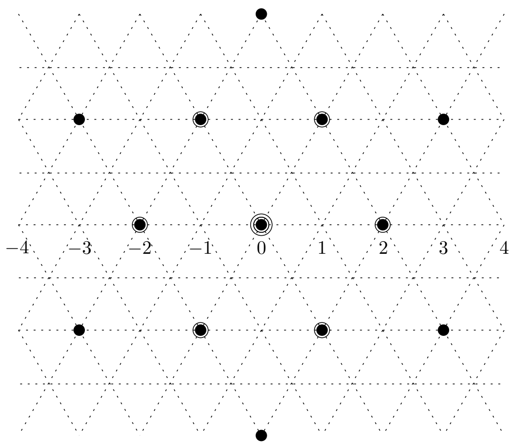

Figure 10: $\Lambda^2\mathrm{Im}(\mathbb{O})$

This diagram is clearly a superposition of the previous two and therefore, we see that $\Lambda^2\mathrm{Im}(\mathbb{O}) \cong \mathrm{Im}(\mathbb{O}) \oplus \mathfrak{g}_2$.

Remark 2.24. One can also do something similar for $\Lambda^3\mathrm{Im}(\mathbb{O})$. However, in that case one need to know what the "next smallest" irreducible representation of $\mathfrak{g}_2$ is. Even though there is an algorithm to compute that and possibly bigger irreducible representations. The counting arguments needed are a little bit too involved to be included in this workshop. So, we will skip that.

Fact 2.25. From above we see that, $\Lambda^2\mathrm{Im}(\mathbb{O})$ has a 7-dimensional and a 14-dimensional irreducible summands. $\Lambda^3\mathrm{Im}(\mathbb{O})$ contains a 1-dimensional, a 7-dimensional and a 27-dimensional irreducible summands. By using hodge star and the fact that $\mathrm{G}_2 \leq \mathrm{SO}(7)$, we get $\Lambda^4\mathrm{Im}(\mathbb{O}) \cong \Lambda^3\mathrm{Im}(\mathbb{O})$, $\Lambda^5\mathrm{Im}(\mathbb{O}) \cong \Lambda^2\mathrm{Im}(\mathbb{O})$ and $\Lambda^6\mathrm{Im}(\mathbb{O}) \cong \mathrm{Im}(\mathbb{O})$. So, we know the complete decomposition of the exterior algebra $\Lambda\mathrm{Im}(\mathbb{O})$.

The following more explicit description can be found in [Bry03].

Fact 2.26.

$$\begin{aligned}
\Lambda^2_7 V^* &= \left\{\alpha \in \Lambda^2 V^* \mid \alpha \wedge \varphi = 2 * \alpha\right\} \\
\Lambda^2_{14} V^* &= \left\{\alpha \in \Lambda^2 V^* \mid \alpha \wedge \varphi = - * \alpha\right\} \\
\Lambda^3_1 V^* &= \mathbb{R}\langle\varphi\rangle \\
\Lambda^3_7 V^* &= \left\{* (\alpha \wedge \varphi) \mid \alpha \in \Lambda^1 V^*\right\} \\
\Lambda^3_{27} V^* &= \left\{\alpha \in \Lambda^3 V^* \mid \alpha \wedge \varphi = 0 \text{ and } \alpha \wedge *\varphi = 0\right\}
\end{aligned}$$

where $V = \mathrm{Im}(\mathbb{O})$ and $\Lambda^p_d V^*$ represents the d-dimensional irreducible summand of $\Lambda^p\mathrm{Im}(\mathbb{O})^*$. (Note that $V \cong V^*$ as G_2-representations.)

3 Topology and Geometry

3.1 Topology of G_2-manifolds

Given a smooth n-manifold M, one can consider its frame bundle F. A fiber F_p of F over a point $p \in M$ consists of all linear isomorphisms $L : \mathbb{R}^n \to T_pM$ and is naturally diffeomorphic to $\mathrm{GL}(n, \mathbb{R})$. However, there is no distinguished identity element. $\mathrm{GL}(n, \mathbb{R})$ naturally acts on F from the right by precomposition. This action is free and transitive on fibers.

Definition 3.1. Let $G \leq \mathrm{GL}(n, \mathbb{R})$. We say M has a G-structure if F admits a subbundle on whose fibers G acts freely and transitively. (Here, we restrict the action of $\mathrm{GL}(n, \mathbb{R})$ to G.)

Example 3.2.
1. $O(n)$-structure is a Riemannian metric.
2. $U(n)$-structure is an almost complex structure with hermitian metric.
3. $Sp(n)$-structure on a $4n$-manifold is an almost hyperKahler structure.

Indeed, if a manifold has an $O(n)$-structure, then for a given $p \in M$ we may pullback the standard metric g_0 on $\mathbb{R}^n$ to T_pM using any of the frames. Let L and T be two frames at p and $A \in O(n)$ such that $L = TA$. Then $L^{-1} = A^{-1}T^{-1}$. Therefore,

$$L^{-1*}g_0 = (A^{-1}T^{-1})^* g_0 = T^{-1*}A^{-1*}g_0 = T^{-1*}g_0.$$

Conversely, given a Riemannian manifold (M, g) one can restrict to frames which are isometries. This gives us an $O(n)$-structure.

Definition 3.3. An almost G_2-manifold is a 7-manifold with a G_2-structure.

A very similar observation shows us that almost G_2-manifolds admit a global section of positive 3-forms. Since $\mathrm{G}_2 \leq \mathrm{SO}(7)$, one can also pullback the standard metric and orientation on $\mathbb{R}^7$. These structures we pullback are the same ones given by Proposition 1.41.

Exercise 3.4. Let M be a 7-manifold with a global 3-form φ which is (point-wise) positive. Prove that M has a G_2-structure.

Remark 3.5. So, once there is a global positive 3-form φ on a 7-manifold M, it is naturally an oriented Riemannian manifold. Furthermore, it admits a cross product structure on each of its tangent spaces.

Remark 3.6. Similarly, one can define an almost Spin(7)-manifold to be an 8-manifold which has a Spin(7)-structure. Again, such manifolds will admit a Riemannian metric, a volume form and a triple cross product operation.

Proposition 3.7. An almost G_2-manifold is spin.

Proof. Follows immediately from the fact that $\mathrm{G}_2 \leq \mathrm{Spin}(7)$. □

Proposition 3.8. A spin 7-manifold M admits a G_2-structure.

Proof. Since M is spin, its SO(7)-frame bundle has a 2-1 covering P whose fibers are homeomorphic to Spin(7). Recall that $\mathrm{Spin}(7)/\mathrm{G}_2 \cong S^7$. So, finding a global section of G_2 cosets in the Spin(7) bundle is equivalent to finding a global section of an S^7 bundle. Since the obstructions of finding a section (of cosets) live in $H^{i+1}(M, \pi_i(S^7))$, they all vanish. Since G_2 does not intersect $\ker(\pi : \mathrm{Spin}(7) \to \mathrm{SO}(7))$, this gives us a global section of G_2 cosets of the frame bundle. □

Corollary 3.9. *M has a G_2-structure if and only if $w_1(M) = 0$ and $w_2(M) = 0$.*

3.2 Geometry

Definition 3.10. A 7-manifold (M, φ) with a G_2-structure is called a G_2-manifold if $\nabla\varphi = 0$ where ∇ is the Levi-Civita connection of the metric g_φ induced from φ. In this case, we sometimes say φ is integrable or torsion free.

Remark 3.11. This is equivalent to Riemannian holonomy group $Hol(g_\varphi)$ being a subgroup of G_2.

Theorem 3.12 ([FG82]). (M, φ) is a G_2-manifold if and only if

$$d\varphi = 0 \text{ and } d *_\varphi \varphi = 0.$$

Remark 3.13. For compact M, φ is harmonic if and only if it is parallel.

Example 3.14. $\mathbb{R}^7, T^7, K3 \times T^3, CY^3 \times S^1$ are examples of G_2-manifolds.

On $K3 \times T^3$, one can explicitly write down

$$\varphi = dx^{123} - \sum_{i=1}^{3} dx^i \wedge w_i$$

where (x_i) are the coordinates on T^3 and (w_i) are the symplectic structures of the K3 surface.

On $CY^3 \times S^1$,

$$\varphi = \mathrm{Re}(\Omega) - dt \wedge w$$

where t is the coordinates on S^1, ω is the symplectic structure and Ω is the holomorphic volume form on CY^3.

However, in all these examples the holonomy group is a proper subgroup of G_2.

Example 3.15. First example with holonomy equal to G_2 is constructed by Bryant in 1987. It is an open and incomplete example. Then, Bryant and Salamon constructed a complete example in 1989 (still non-compact). Later, Joyce constructed the first compact example in 1996.

3.3 Associative, coassociative and Cayley submanifolds

3.3.1 Point-wise aspects

Definition 3.16. Let (V, g) be an n-dimensional inner product space with a k-form $\beta \in \Lambda^k V^*$. If β satisfies,

$$|\beta(e_1, \cdots, e_k)| \leq 1$$

for all orthonormal $e_1, \cdots, e_k \in V$, then β is called a calibration form and (V, g, β) is called a calibrated space.

Exercise 3.17. Prove that the associate calibration φ, its Hodge dual $*\varphi$ and Cayley calibration Φ forms are calibration forms.

Definition 3.18. Let (V, g, β) be a calibrated space. Then a k-plane Λ is called calibrated if it has an orthonormal basis $\{e_1, \cdots, e_k\}$ such that $\beta(e_1, \cdots, e_k) = 1$.

Remark 3.19. A calibrated 3-plane in $(\mathrm{Im}(\mathbb{O}), \varphi)$ is called an associative plane. The space of all associative planes is called the associative Grassmannian. G_2 acts transitively on this space with the stabilizer group SO(4).

A calibrated 4-plane in $(\mathrm{Im}(\mathbb{O}), *\varphi)$ is called an coassociative plane. The space of all coassociative planes is called the coassociative Grassmannian. This space is isomorphic to the associative Grassmannian.

A calibrated 4-plane in $(\mathbb{O}, \Phi)$ is called a Cayley plane. The space of all Cayley planes is called the Cayley Grassmannian. Spin(7) acts transitively on the Cayley Grassmannian with stabilizer group

$$(\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2))/\pm 1$$

In fact, the Cayley Grassmannian is also isomorphic to $Gr(3, 7)$.

3.3.2 Global aspects

Definition 3.20 ([HL82])**.** Let (M, g) be a Riemannian manifold. If a closed k-form β restricts to a calibration form at every tangent space then β is called a calibration form and (M, g, β) is called a calibrated manifold.

Definition 3.21. Let (M, g, β) be a calibrated manifold. Let X be a compact oriented submanifold of M such that $Vol_X = \beta|_X$. Then X is called a calibrated submanifold.

Remark 3.22. Calibrated submanifolds are volume minimizing in their homology class. Indeed, let X be a calibrated submanifold and X' be another compact submanifold in the homology class of X. Then,

$$Vol(X) = \int_X \beta = \int_{X'} \beta \leq \int_{X'} Vol_{X'} = Vol(X').$$

Remark 3.23. 3-submanifolds and 4-submanifolds of a G_2-manifold, calibrated with respect to φ and $*\varphi$ are called associative and coassociative submanifolds. Calibrated submanifolds of a Spin(7)-manifold are called Cayley submanifolds.

4 Complex G_2-manifolds and Connections with Symplectic Topology

In this section, we usually work over the complex numbers and the objects that appeared before may need to be complexified e.g. $\mathbb{O}, \varphi$,etc.. Main reference is [AY18].

4.1 G_2-spaces

Definition 4.1. $\mathrm{G}_2^{\mathbb{C}}$ is the automorphism group of the complexified octonions $\mathbb{O} = \mathbb{O} \otimes \mathbb{C}$.

Proposition 4.2.

$$\mathrm{G}_2^{\mathbb{C}} = \{A \in \mathrm{SL}(\mathrm{Im}(\mathbb{O})) \mid A\varphi = \varphi\}$$

Definition 4.3. We call the quadruple (V, φ, Ω, B) satisfying

$$(\iota(u)\varphi) \wedge (\iota(v)\varphi) \wedge \varphi = 6B(u, v)\Omega \tag{7}$$

and $N(\Omega) = 1$ (where N is the quadratic form associated to B) a $\mathrm{G}_2^{\mathbb{C}}$-(vector) space.

Remark 4.4. One can also define real G_2-spaces in a similar manner. In fact, over $\mathbb{R}$, specifying a positive φ is enough.

4.2 Complexification of a G_2-space

Given (V, φ), it determines a (real) G_2-space (V, φ, Ω, g). Let $V_{\mathbb{C}} = V \oplus iV$. Furthermore, we can extend all of the structures complex linearly and the equation (7) continues to hold. This implies that the complexified three-form is still non-degenerate. Therefore, we get a (complex) G_2-space $(V_{\mathbb{C}}, \varphi_{\mathbb{C}}, \Omega_{\mathbb{C}}, g_{\mathbb{C}})$.

We can also extend g as a hermitian form h. Explicitly, we define

$$h(x + iy, z + iw) = g(x, z) + g(y, w) + i\left(g(y, z) - g(x, w)\right).$$

Then, the real part of h is a positive definite metric and the imaginary part is a symplectic form ω on $V_{\mathbb{C}}$.

Remark 4.5. If V is a half dimensional subspace of W with an almost complex structure J such that $V \oplus JV = W$, we could use J in place of i in the above construction. This flexibility will be important later on.

4.3 Compatible structures on a $G_2^{\mathbb{C}}$-space

Definition 4.6. We say that the triple $(g, \omega, \varphi_{\mathbb{C}})$ is compatible if there is a real 7 dimensional subspace Λ of V and a positive φ on Λ (determining a metric g' on Λ) such that

1. $V = \Lambda \oplus J\Lambda =: \Lambda_{\mathbb{C}}$
2. $\varphi_{\mathbb{C}}$ is the complex linear extension of φ
3. $g + i\omega$ is the hermitian extension of g'.

In this case, we say they are induced from (Λ, φ, J).

Remark 4.7. Basically all of this definition is to say that we consider $(g, \omega, \varphi_{\mathbb{C}})$ compatible if they arise from complexification of a G_2-space.

Remark 4.8. Of course, this resembles the Kahler picture of (ω, g, J).

$$\mathrm{GL}(n, \mathbb{C}) \cap \mathrm{O}(2n) = \mathrm{O}(2n) \cap \mathrm{Sp}(2n) = \mathrm{Sp}(2n) \cap \mathrm{GL}(n, \mathbb{C}) = \mathrm{U}(n), \tag{8}$$

Fact 4.9 ([AY18])**.**

$$G_2^{\mathbb{C}} \cap \mathrm{O}(14) = G_2^{\mathbb{C}} \cap Sp(14) = G_2$$

4.4 Isotropic Associative Grassmannian

Definition 4.10. The associator bracket is defined as follows

$$[u, v, w] = u \times (v \times w) + B(u, v)w - B(u, w)v.$$

Definition 4.11. Let L be a (real) 3-dimensional subspace of $\mathrm{Im}\mathbb{O} = \mathbb{C}^7$. We call L isotropic associative if

1. $\omega\big|_L \equiv 0$, and
2. $[u, v, w]_{\mathbb{C}} = 0$ for $u, v, w \in L$.

We denote the space of all isotropic associative planes by $I_3^{\varphi} \subset Gr^{\mathbb{R}}(3, 14)$.

The following lemma describes the tangent space of I_3^{φ} in $Gr^{\mathbb{R}}(3, 14)$ at a (real) associative plane.

Lemma 4.12. Let $L = \langle e_1, e_2, e_3 \rangle$ be an associative plane in $\mathbb{R}^7$. Then its natural embedding in $\mathbb{C}^7 = \mathrm{Im}\mathbb{O}$ is an isotropic associative. Denote the (real) tautological bundle over $Gr^{\mathbb{R}}(3, 14)$ by $\mathbb{E}$. Also, set $\mathbb{V} = \mathbb{E}^{\perp B}$. Then $\mathbb{E}^{\perp g} = J\mathbb{E} \oplus \mathbb{V}$ and

$$\begin{aligned} T_L I_3^{\varphi} \quad = \quad & \Big\{ \sum_{i=1}^{3} e^i \otimes (f_i + v_i) \in \mathbb{E}^* \otimes_{\mathbb{R}} (J\mathbb{E} \oplus \mathbb{V}) \\ & \quad \mid \sum e_i \times v_i = 0 \text{ and } \omega(e_i, f_j) = \omega(e_j, f_i) \Big\} \end{aligned} \tag{9}$$

4.5 $G_2^{\mathbb{C}}$-manifolds

Definition 4.13. A (real) 14-dimensional manifold M is called an (almost) $G_2^{\mathbb{C}}$-manifold if has a $G_2^{\mathbb{C}}$-structure.

Proposition 4.14. A $G_2^{\mathbb{C}}$-manifold M naturally has the following structures

- an almost complex structure $J \in \Gamma(M; \mathrm{End}(TM))$

- a $\mathbb{C}$-linear three-form $\varphi \in \Omega^3(M;\mathbb{C})$
- a $\mathbb{C}$-linear seven-form $\Omega \in \Omega^7(M;\mathbb{C})$
- a symmetric bilinear form $B \in \Gamma(M; \mathrm{S}^2(TM) \otimes \mathbb{C})$

Example 4.15. We start with a (real) 7-dimensional G_2 manifold (M, φ) and we think of g as an isomorphism between TM and T^*M. Using this isomorphism, we think of φ as an element of $\Gamma(\Lambda^3 TM)$. Therefore, (T_p^*M, φ) is a G_2-space. The vertical subspace of $T_\alpha T^*M$ is canonically defined and isomorphic to $T^*_{\pi(\alpha)}M$. The vertical subbundle defines a Lagrangian 7-plane distribution on $(T^*M, \omega_{\text{can}})$. The space of compatible almost complex structures on $(T_\alpha T^*M, \Lambda = T^*_{\pi(\alpha)}M, \varphi, \omega_{\text{can}})$ is contractible [AY18]. Therefore, one can find a global almost complex structure J such that the complexification of (Λ, φ) with respect to J gives us a compatible triple $(\omega_{\text{can}}, \varphi_{\mathbb{C}}, g)$. Compatibility here means compatibility at every point.

4.6 Deformations of associative submanifolds

Theorem 4.16. Let (Y, s) be a closed oriented 3-manifold with a *Spin*^{c} structure, and $(Y, s) \subset (M, \varphi)$ be an imbedding as an associative submanifold of some G_2 manifold (note that such imbedding always exists). Then the isotropic associative deformations of (Y, s) in the complexified G_2 manifold $M_{\mathbb{C}}$ is given by Seiberg-Witten equations :

$$\begin{aligned} \not{D}_{\mathbf{A}}(x) &= 0 \\ *F_A &= \sigma(x). \end{aligned}$$

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